Final Test Review

5a) $\chi = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$, find $\langle S_x\rangle$, $\langle S_y\rangle$, $\langle S_z\rangle$

- $\langle S_x\rangle = \langle\chi|S_x|\chi\rangle = \frac{1}{\sqrt{5}}(1\ \ -2i)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$

$\frac{\hbar}{10}(-2i.\ 1)\begin{pmatrix}1\\2i\end{pmatrix} = \frac{\hbar}{10}(-2i+2i) = \boxed{0 = \langle S_x\rangle}$

- $\langle S_y\rangle = \langle\chi|S_y|\chi\rangle = \frac{1}{\sqrt{5}}(1\ \ -2i)\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$

$= \frac{\hbar}{10}(-2i^2\ \ -i)\begin{pmatrix}1\\2i\end{pmatrix} = \frac{\hbar}{10}(2\ \ -i)\begin{pmatrix}1\\2i\end{pmatrix} = \frac{\hbar}{10}(2-2i^2)$

$= \frac{\hbar}{10}(2+2) = \frac{4\hbar}{10} = \boxed{\frac{2\hbar}{5} = \langle S_y\rangle}$

- $\langle S_z\rangle = \langle\chi|S_z|\chi\rangle = \frac{1}{\sqrt{5}}(1\ \ -2i)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$

$= \frac{\hbar}{10}(1\ \ 2i)\begin{pmatrix}1\\2i\end{pmatrix} = \frac{\hbar}{10}(1+4i^2) = \frac{\hbar}{10}(1-4) = \boxed{-\frac{3\hbar}{10} = \langle S_z\rangle}$

- When measuring spin S_x along $x \Rightarrow$ prob. of meas. $\hbar/2$

$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow \frac{\hbar}{2}\begin{pmatrix}-\lambda&1\\1&-\lambda\end{pmatrix} = \frac{\hbar}{2}(\lambda^2-1) = 0$

$\lambda = \pm\hbar/2$

so $\lambda = \hbar/2, -\hbar/2$

for $\hbar/2$:

$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}a\\b\end{pmatrix}$ $\quad b = \hbar/2\, a$, $a = \hbar/2\, b$ $\Rightarrow a = b$ hence $\begin{pmatrix}1\\1\end{pmatrix} = \chi_x$

so $(1\ \ 1)\begin{pmatrix}1\\1\end{pmatrix} = 1+1 = 2$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \chi_x$

Prob. of $\hbar/2$?

$P = |\langle\chi|\chi_x\rangle|^2 = \left|\frac{1}{\sqrt{5}}(1\ \ -2i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{10}}(1-2i)\right|^2$

$= \left(\frac{1}{\sqrt{10}}\right)^2(1+2i)\begin{pmatrix}1\\-2i\end{pmatrix} = \frac{1}{10}(1-4i^2) = \frac{5}{10} = \boxed{\frac{1}{2}}$

5b) $\hat{H} = -\gamma B_0 \hat{S}_z$ $\quad E = ?$

$H\psi = E\psi \Rightarrow \hat{H} = -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$

$$\boxed{\begin{aligned} E_+ &= -\gamma B_0 \hbar/2 \\ E_- &= \gamma B_0 \hbar / 2 \end{aligned}}$$

$|c_n|^2 \Rightarrow$ Prob of E_n

$$\boxed{\begin{aligned} |c_\uparrow|^2 &= \tfrac{1}{5} \\ |c_\downarrow|^2 &= 4/5 \end{aligned}}$$

$P = |\langle \chi_1 | \chi_2 \rangle|^2$

$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$

$\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $\left|\frac{1}{\sqrt{5}}(1 \; -2i)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{5}}(1+0)\right|^2 = \boxed{1/5}$
- $\left|\frac{1}{\sqrt{5}}(1 \; -2i)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{5}}(0-2i)\right|^2 = \frac{1}{5}(4) = \boxed{\frac{4}{5}}$

c) Ox = 8 e^- explain Hund's rules

1.) The highest total spin will have the lowest energy
2.) The highest total orbital angular momentum will have the lowest energy
3.) If a shell is no more than 1/2 filled, $J = |L - S|$
If a shell is more than 1/2 filled, $J = |L + S|$

$\boxed{(1s)^2 (2s)^2 (2p)^4}$

1s $\uparrow\downarrow$
2s $\uparrow\downarrow$
2p $\downarrow\uparrow$ $\quad \uparrow \quad \uparrow$
$m=1 \quad m=0 \quad m=-1$

$\vec{L} = ?$: $1 + 1 + 0 - 1 = \boxed{1 = L}$

$\vec{S} = ?$: $1/2 + 1/2 + 1/2 - 1/2 = \boxed{1 = S}$

Since more than, $J = |L + S| = |1 + 1| = \boxed{2 = J}$

S p d f
0 1 2 3

$^{2S+1}L_J \; : \; ^{2(1)+1}P_2 = \boxed{^3P_2}$

d) In a solid, what are valence e^-?
The most loosely bound outermost electrons & no longer subject to the Coulomb field, but rather to the potential of the lattice.

$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \Longrightarrow$

5d) $E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3}$ $\rho = \frac{N}{V}q$

$$8.96\,\frac{g}{cm^3}\left(\frac{1\ mol}{63.5\,g}\right)\left(\frac{6.022\times10^{23}\,at}{1\ mol}\right)\left(\frac{(100)^3\,cm^3}{1\ m^3}\right) = 8.497\times10^{28}\ \frac{atoms}{m^3}$$

$$\rho = \frac{N}{V}\cdot q = 8.497\times10^{28}(1.2) = 1.0196\times10^{29}\ \frac{atoms}{m^3}$$

$$E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3} = \frac{(1.05\times10^{-34})^2}{2(9.11\times10^{-31})}\left(3(1.0196\times10^{29})\pi^2\right)^{2/3}$$

$$E_F = 1.263\times10^{-18}\ J\left(\frac{1\,eV}{1.6\times10^{-19}\,J}\right) = \boxed{7.8997\ eV}$$

The Fermi energy represents the energy of the highest occupied quantum state for a state of fermions