Problem 3

Let's consider a particle in the harmonic oscillator: $V = \frac{1}{2}m\omega^2 x^2$.

The initial wave function is $\Psi(x, 0) = \frac{1}{\sqrt{5}}(\psi_0 + 2\psi_1)$, where ψ_n are the eigenstates of the Hamiltonian.

a) What is the wave function $\Psi(x, t)$ at later time?

Because the potential V is time-independent, the wave functions are separable, i.e. $\Psi(x, t) = \psi(x)T(t)$. The time part is just the exponential of the energy:

$T(t) = e^{\frac{-iE_n t}{\hbar}} = e^{-\omega(n+1/2)t}$.

(Recall that, for the harmonic oscillator, $E_n = \hbar\omega(n + 1/2)$). Then

$\Psi(x, t) = \sum \psi_n(x)T_n(t) = \sum \psi_n(x)e^{-i(n+1/2)wt}$.

$\Psi(x, t) = \frac{1}{\sqrt{5}}(\psi_0 e^{-\frac{i\omega t}{2}} + 2\psi_1 e^{-\frac{3i\omega t}{2}})$

b) Calculate $\langle x\rangle$, $\langle p\rangle$ at a time t. Do they depend on time?

When we learned about the harmonic oscillator, we had to write out the integrals to calculate these quanities:

$\langle\hat{x}\rangle = \int_{-\infty}^{\infty} \Psi^*(x, t)\hat{x}\Psi(x, t)dx$, $\langle\hat{p}\rangle = \int_{-\infty}^{\infty} \Psi^*(x, t)\hat{p}\Psi(x, t)dx$.

After going through chapter 3, the work will be easier in the Dirac notation: $|\Psi\rangle = \frac{1}{\sqrt{5}}\left(e^{-\frac{i\omega t}{2}}|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}|\psi_1\rangle\right)$

$\langle\hat{x}\rangle = \langle\Psi|\hat{x}\Psi\rangle = \frac{1}{\sqrt{5}}\left(e^{\frac{i\omega t}{2}}\langle\psi_0| + 2e^{\frac{3i\omega t}{2}}\langle\psi_1|\right)\hat{x}\left(\frac{1}{\sqrt{5}}\left(e^{-\frac{i\omega t}{2}}|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}|\psi_1\rangle\right)\right)$.

With the harmonic oscillator, the $\hat{x}$ and $\hat{p}$ operators can be rewritten in terms of the ladder operators (as given on the test):

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+)$, and $\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_+ - \hat{a}_-)$.

Let's rewrite the ket $|\hat{x}\Psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_- + \hat{a}_+)\frac{1}{\sqrt{5}}\left(e^{-\frac{i\omega t}{2}}|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}|\psi_1\rangle\right)$

$= \frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-\frac{i\omega t}{2}}\hat{a}_-|\psi_0\rangle + e^{-\frac{i\omega t}{2}}\hat{a}_+|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}\hat{a}_-|\psi_1\rangle + 2e^{-\frac{3i\omega t}{2}}\hat{a}_+|\psi_1\rangle\right)$.

Remember that $\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}$ and $\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}$. Our ket becomes $|\hat{x}\Psi\rangle = \frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{2m\omega}}\left(0 + e^{-\frac{i\omega t}{2}}\sqrt{1}|\psi_1\rangle + 2e^{-\frac{3i\omega t}{2}}\sqrt{1}|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}\sqrt{2}|\psi_2\rangle\right)$.

Now the expectation value of x becomes:

$\langle\hat{x}\rangle = \frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{\frac{i\omega t}{2}}\langle\psi_0| + 2e^{\frac{3i\omega t}{2}}\langle\psi_1|\right)\left(e^{-\frac{i\omega t}{2}}\sqrt{1}|\psi_1\rangle + 2e^{-\frac{3i\omega t}{2}}\sqrt{1}|\psi_0\rangle + 2e^{-\frac{3i\omega t}{2}}\sqrt{2}|\psi_2\rangle\right)$

Because the wave functions of the harmonic oscillator are orthonormal, i.e. $\langle\psi_n \mid \psi_m\rangle = \delta_{nm}$, we can neglect any inner products of different wave functions. The only terms that remain in the expectation value of x are those involving $\langle\psi_0|\psi_0\rangle = 1$ and $\langle\psi_1|\psi_1\rangle = 1$

$\langle\hat{x}\rangle = \frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}\left(2e^{\frac{i\omega t - 3i\omega t}{2}}\langle\psi_0|\psi_0\rangle + 2e^{\frac{3i\omega t - i\omega t}{2}}\langle\psi_1|\psi_1\rangle\right) = \frac{1}{5}\sqrt{\frac{\hbar}{2m\omega}}2\left(e^{-i\omega t} + e^{i\omega t}\right) = \frac{4}{5}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$.

We could do the same process to calculate $\langle \hat{p} \rangle$, or we could remember that $\langle \hat{p} \rangle = m\frac{d\langle \hat{x} \rangle}{dt} = m\frac{d}{dt}\left(\frac{4}{5}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right)$

$= -m\frac{4}{5}\sqrt{\frac{\hbar}{2m\omega}}\omega\sin(\omega t) = -\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t)$

Yes, they both depend on time.

c) Check the Ehrenfest theorem for this particle: $\frac{d\langle \hat{p} \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$

$\frac{d\langle \hat{p} \rangle}{dt} = \frac{d}{dt}\left(-\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\sin(\omega t)\right) = -\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t)$

$\left\langle -\frac{\partial V}{\partial x} \right\rangle = \left\langle \Psi \right| - \frac{\partial V}{\partial x}\Psi \rangle$

$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}m\omega^2 x^2\right) = m\omega^2 x.$

Then $\left\langle -\frac{\partial V}{\partial x} \right\rangle = \left\langle \Psi \right| - m\omega^2 x\Psi \rangle = -m\omega^2 \langle \hat{x} \rangle$**. We know** $\langle \hat{x} \rangle$ **from part (c).** $\left\langle -\frac{\partial V}{\partial x} \right\rangle = -m\omega^2\frac{4}{5}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = -\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) = \frac{d\langle \hat{p} \rangle}{dt}$. Check! ✓

d) Show in general $[f(x), \hat{p}] = i\hbar\frac{df}{dx}$

$[f(x), \hat{p}]\, g(x) = (f(x)\hat{p} - \hat{p}f(x))\, g(x) = f(x)\left(-i\hbar\frac{d}{dx}\right)g(x) - -i\hbar\frac{d}{dx}\left(f(x)g(x)\right)$

$= -i\hbar f(x)\frac{dg(x)}{dx} + i\hbar\frac{df(x)}{dx}g(x) + i\hbar\frac{g(x)}{dx}f(x) = i\hbar\frac{df(x)}{dx}g(x)$

Then $[f(x), \hat{p}] = i\hbar\frac{df(x)}{dx}$

e) Use the equation of motion for the operator p to confirm the Ehrenfest theorem.

As given on the test, the Heisenberg equation of motion is:

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{Q}\right] \right\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

For our purposes, $\hat{Q} = \hat{p}$. Because $\hat{p} = -i\hbar\frac{d}{dt}$ is time-independent, $\left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle = 0.$

$\left[\hat{H}, \hat{Q}\right] = \left[\left(-\frac{\hat{p}^2}{2m} + V(x)\right), \hat{p}\right] = \left[-\frac{\hat{p}^2}{2m}, \hat{p}\right] + [V(x), \hat{p}].$

$\hat{p}^2$commutes with $\hat{p}$, so $\left[\hat{H}, \hat{p}\right] = [V(x), \hat{p}].$

From part (e),

$[V(x), \hat{p}(x)] = i\hbar\frac{d}{dx}\left(V(x)\right) = i\hbar\frac{d}{dx}\left(\frac{1}{2}m\omega^2 x^2\right) = i\hbar m\omega^2 x.$

$\left\langle \left[\hat{H}, \hat{p}\right] \right\rangle = \left\langle i\hbar m\omega^2 x \right\rangle = i\hbar m\omega^2 \langle x \rangle$

We know $\langle x \rangle$ from part (b), so we have:

$\left\langle \left[\hat{H}, \hat{p}\right] \right\rangle = i\hbar m\omega^2\frac{4}{5}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = i\hbar\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t)$

Now we can write the equation of motion: $\frac{d\langle\hat{p}\rangle}{dt} = \frac{i}{\hbar} i\hbar \frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t) = -\frac{4}{5}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t)$

Just like Ehrenfest's Theorem! ✓