Problem 3

An electron is in the spin state χ_0=A$\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ in the basis (χ_+,χ_-) of eigenspinors of S_z

a) Find the Normalization constant A

To normalize χ_0, set its inner product to 1:
$\langle\chi_0|\chi_0\rangle = 1$
$\chi_0^H\chi_0 = 1$ where the H means the hermitian, or you take the conjugate transpose of the vector.
$A^* \begin{bmatrix} 1 & -2i \end{bmatrix} \cdot A \begin{bmatrix} 1 \\ 2i \end{bmatrix} = 1$
$|A|^2 (1+4) = 1$
$A = \frac{1}{\sqrt{5}}$

b) Using Pauli matrices, find the expectation values $\langle S_x\rangle$,$\langle S_y\rangle$ and $\langle S_z\rangle$

The Pauli matrices are $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

The spin operators are $S_x = \frac{\hbar}{2}\sigma_x$, $S_x = \frac{\hbar}{2}\sigma_x$, and $S_x = \frac{\hbar}{2}\sigma_x$

$$\langle S_x\rangle = \langle\chi_0\,|S_x|\,\chi_0\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2i \end{bmatrix}\cdot\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$= \frac{\hbar}{2\cdot 5}\begin{bmatrix} -2i & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \frac{\hbar}{10}(-2i+2i) = 0$$

$$\langle S_y\rangle = \langle\chi_0\,|S_y|\,\chi_0\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2i \end{bmatrix}\cdot\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$= \frac{\hbar}{2\cdot 5}\begin{bmatrix} 2 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \frac{\hbar}{10}(2+2) = \frac{2}{5}\hbar$$

$$\langle S_z\rangle = \langle\chi_0\,|S_z|\,\chi_0\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2i \end{bmatrix}\cdot\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$= \frac{\hbar}{2\cdot 5}\begin{bmatrix} 1 & 2i \end{bmatrix}\begin{bmatrix} 1 \\ 2i \end{bmatrix} = \frac{\hbar}{10}(1-4) = \frac{3}{10}\hbar$$

c) Calculate $\langle S_x^2\rangle$,$\langle S_y^2\rangle$ and $\langle S_z^2\rangle$, and show that their sum is equal to what is expected for $\langle S^2\rangle$

We could go through the process of evaluating each expectation value the long way:

$\left\langle S_x^2 \right\rangle = \langle \chi_0 \left| S_x \cdot S_x \right| \chi_0 \rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2i \end{bmatrix} \cdot \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$

and so forth. Or we could note that

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is hermitian, so $\sigma_x^T = \sigma_x$

and it's orthogonal, so $\sigma_x^{-1} = \sigma_x^T$.

Then $S_x \cdot S_x = \frac{\hbar^2}{4} \sigma_x \cdot \sigma_x^{-1} = \frac{\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\hbar^2}{4} I_2$

Then $\left\langle S_x^2 \right\rangle = \langle \chi_0 \left| S_x \cdot S_x \right| \chi_0 \rangle = \frac{\hbar^2}{4} \langle \chi_0 \left| I_2 \right| \chi_0 \rangle = \frac{\hbar^2}{4} \langle \chi_0 | \chi_0 \rangle = \frac{\hbar^2}{4}$

Similarly, $S_y$and$S_z$are orthogonal hermitian matrices, so

$S_y^2 = S_z^2 = \frac{\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $\left\langle S_x^2 \right\rangle + \left\langle S_y^2 \right\rangle + \left\langle S_z^2 \right\rangle = 3 \left\langle S_x^2 \right\rangle = \frac{3\hbar^2}{4}$

But $S^2 = \frac{3\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\left\langle S^2 \right\rangle = \frac{3\hbar^2}{4}$

and $\left\langle S_x^2 \right\rangle + \left\langle S_y^2 \right\rangle + \left\langle S_z^2 \right\rangle = \left\langle S^2 \right\rangle$

d) When measuring S_z on the state χ_0, what is the probability of measuring $\frac{\hbar^2}{2}$?

This probability is the square of the coefficient of χ_+ .

Recall that $\chi_0 = \frac{1}{\sqrt{5}} \chi_+ + \frac{2i}{\sqrt{5}} \chi_+$, or $\chi_0 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} \end{bmatrix}$

Then the probability of getting $+\frac{\hbar}{2}$ is $\left(\frac{1}{\sqrt{5}} \right)^2 = \frac{1}{5}$

e) When measuring S_x on the state χ_0, what is the probability of measuring $\frac{\hbar}{2}$?

Getting the Eigenspinors of S_x Again, the probability is the square of the coefficient of χ_+ . However, here we need the coefficient of $\chi_+^{(x)}$. In other words, we need to rewrite χ_0 in terms of the eigenspinors of S_x, whereas until now we've been working in the basis of $\left\{ \chi_+^{(z)}, \chi_-^{(z)} \right\}$, the eigenspinors of S_z.

First, then, we need to determine the eigenspinors of S_x.

$S_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so the eigenvalues are determined by setting the determinant of the matrix $S_x - \lambda I_2 = \begin{bmatrix} -\lambda & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\lambda \end{bmatrix}$ to 0.

We then get $-\lambda(-\lambda) - \frac{\hbar^2}{4} = 0$ so $\lambda^2 = \frac{\hbar^2}{4}$ or $\lambda = \pm \frac{\hbar}{2}$.

The eigenvectors come from solving the equation$(S_x - \lambda I_2) \chi_\lambda = 0$ for each eigenvalue.

For $\lambda = +\frac{\hbar}{2}$, we get $\begin{bmatrix} -\frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & -\frac{\hbar}{2} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which reduces to $-\chi_1 + \chi_2 = 0$ or $\chi_1 = \chi_2$. Our first eigenspinor, corresponding to $\lambda = \frac{\hbar}{2}$, is $\chi_+^{(x)} =$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalized, we get $\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -\frac{\hbar}{2}$, we get $\begin{bmatrix} \frac{\hbar}{2} & \frac{\hbar}{2} \\ \frac{\hbar}{2} & \frac{\hbar}{2} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Our next eigenspinor, corresponding to $\lambda = -\frac{\hbar}{2}$, is $\chi_-^{(x)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The Hard Way Now we can consider changing bases from $\left\{\chi_+^{(z)}, \chi_-^{(z)}\right\}$ to $\left\{\chi_+^{(x)}, \chi_-^{(x)}\right\}$. We get the equation $\chi_0 = a \cdot \chi_+^{(z)} + b \cdot \chi_-^{(z)} = c \cdot \chi_+^{(x)} + d \cdot \chi_-^{(x)}$ where a and b are the coefficients given at the beginning, $1/\sqrt{5}$ and $2i/\sqrt{5}$. We know what $\chi_+^{(x)}$ and $\chi_-^{(x)}$ are in terms of the eigenspinors of S_z, so we can rewrite our equation as $\chi_0 = a \cdot \chi_+^{(z)} + b \cdot \chi_-^{(z)} = c \cdot \left(\frac{1}{\sqrt{2}}\chi_+^{(z)} + \frac{1}{\sqrt{2}}\chi_-^{(z)}\right) + d \cdot \left(\frac{1}{\sqrt{2}}\chi_+^{(z)} - \frac{1}{\sqrt{2}}\chi_-^{(z)}\right)$
$= \left(\frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}}\right)\chi_+^{(z)} + \left(\frac{c}{\sqrt{2}} - \frac{d}{\sqrt{2}}\right)\chi_-^{(z)}$.

If we write this in matrix form, we get the equation

$$\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{c}{\sqrt{2}} & \frac{c}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}.$$

Note that the matrix here is unitary - the columns have norm 1 and are orthogonal - and is hermitian. Then this matrix is its own inverse, so

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1+2i}{\sqrt{10}} \\ \frac{1-2i}{\sqrt{10}} \end{bmatrix}.$$

The probability of measuring $\frac{\hbar}{2}$with $S_x$is $\left|c^2\right| = \left|\frac{1+2i}{\sqrt{10}}\right|^2 = 1/10(1+4) = 1/2$.

The Easier Way Instead of going through the hassle of changing bases, we could realize that all we want is the coefficient of $\chi_+^{(x)}$. We can get this by projecting $\chi_0$onto $\chi_+^{(x)}$.

The projection formula is $\frac{\langle \chi_0 | \chi_+^{(x)} \rangle}{\langle \chi_+^{(x)} | \chi_+^{(x)} \rangle} \left| \chi_+^{(x)} \right\rangle$. Because $\left\| \chi_+^{(x)} \right\| = 1$ and we just want the coefficient, we just need the inner product

$\left\langle \chi_0 | \chi_+^{(x)} \right\rangle = \begin{bmatrix} 1/\sqrt{5} & 2i/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1+2i}{\sqrt{10}}$. The probability is the square of the coefficient, $\frac{1}{10}(1+4) = 1/2$.