KC
Eb

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

a) Inside the well $V=0$.

SE: $i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \Psi$ This is a one-dimensional problem so ∇^2 can be written as $\frac{\partial^2}{\partial x^2}$

$$\rightarrow i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi \qquad (1.a)$$

Now, let $\Psi(x,t)$ be separable into two equations

$\Psi = X(x)\,T(t)$. Plugging this into (1.a) above:

$$i\hbar X T_t = \frac{-\hbar^2}{2m} X_{xx} T$$

And dividing both sides by XT:

$$i\hbar \frac{T_t}{T} = \frac{-\hbar^2}{2m} \frac{X_{xx}}{X}$$

The left hand side is a function of t only while the right hand side is a function of x only. This can only mean that both sides are equal to some constant. Let's call that constant "E" just for funzies.

Now $i\hbar \frac{T_t}{T} = E \Rightarrow \frac{dT}{dt} = -\frac{iE}{\hbar} T \Rightarrow T = Ae^{-iEt/\hbar}$

$$-\frac{\hbar^2}{2m} \frac{X_{xx}}{X} = E \Rightarrow \frac{d^2X}{dx^2} = \frac{-2mE}{\hbar^2} X \Rightarrow \quad ?$$

The second one isn't so immediately solveable but if you notice that $\frac{-2mE}{\hbar^2}$ is always negative since E is always positive then you can replace it by a constant that is always negative. One way to gaurantee that your constant is negative is to square it and stick a negative sign out front ↓

$$\frac{d^2X}{dx^2} = -k^2 X \quad \text{and} \quad k = \frac{\sqrt{2mE}}{\hbar}$$

The above equation is much easier to solve since we know that the only function whose second derivative is proportional to the negative of itself is a linear combination of sines and cosines:

$$X = B\cos(kx) + C\sin(kx)$$

Now $\Psi(x,t)$ can be written as: $\Psi(x,t) = \left(B\cos(kx) + C\sin(kx)\right) A e^{-iEt/\hbar}$

→ Our next task is to eliminate those pesky constants that came from integration. We know that at $x=0$ and at $x=a$ $\Psi(x,t)=0$ since the potential is infinite there. Because this is true for all times t: $\Psi(0,t) = \Psi(a,t) = X(0)T(t) = X(a)T(t)$ it must be true that: $X(0) = 0 = X(a)$. That should help us determine the unknown constants:

$$X(0) = 0 = B\cos(0) + C\sin(0) = B \qquad B = 0$$
$$X(a) = 0 = C\sin(ka)$$

Either C or $\sin(ka)$ is zero. If the culprit is C then $B = C = 0$ and $X(x) = 0$! That's not a good solution. So $\sin(ka)$ must be zero. Sine is zero when its argument is an integer multiple of π:

$$0 = \sin(ka) = \sin(n\pi) \quad \text{so} \quad ka = n\pi \qquad n = 0, 1, 2, \ldots$$

Note that if $n=0$ then $k=0$ which would make $X(x)$ always zero again. Let's agree not to let $n=0$ then.

Now $k_n = \frac{n\pi}{a}$ $n = 1, 2, 3, 4, \ldots$ (We'd better change k to k_n since there are so many of them)

Now we have as many $X(x)$'s as we have k's (an infinity of them):

$$X_n(x) = C \sin(k_n x)$$

and we have just as many possible values of the constant E:

$$k_n = \frac{\sqrt{2mE_n}}{\hbar} \quad \text{or} \quad E_n = \frac{\hbar^2 k_n^2}{2m}$$

Before we can say we solved the problem we need to figure out that last constant: $A \cdot C$

$$\Psi(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} A \cdot C \sin(k_n x) e^{-iE_n t/\hbar}$$

The trick this time is simply to rewrite $X_n(x) = A \cdot C \sin(k_n x)$ and recall that this part of the solution needs to satisfy the following relation:

$$1 = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = AC \int_0^a \sin^2(k_n x)\, dx \qquad AC = \sqrt{\frac{2}{a}}$$

Finally we have it! $\Psi(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{a}} \sin(k_n x) e^{-iE_n t/\hbar}$ (*)

where $k_n = \frac{n\pi}{a} \qquad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$ (**)

Notice that the spatial part X_n constitutes the so called "stationary states":

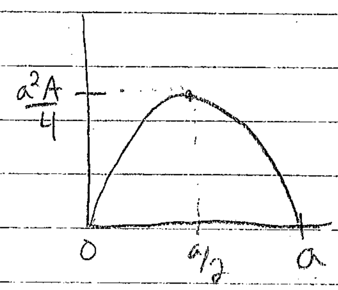
$$X_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

b) $\Psi(x,0) = Ax(a-x) = Aax - Ax^2$

- To sketch this function notice that at $x=a$ and $x=0$ it is 0. Taking the derivative I find:

$$\Psi(x,0)_x = -2Ax + Aa$$

This function is zero when $x = a/2$. This must be the maximum since it's concave down and quadratic ($\Psi_{xx} = -2A$, and the highest order is 2)



$$\Psi(\tfrac{a}{2},0) = -2A\frac{a^2}{2} + A\frac{a^2}{4} = \frac{a^2 A}{4}$$

- We can always express a function as a linear combination of eigenstates when both are in the same space (in our case Hilbert space). So yes.

- To find the coefficients c_n there are two great methods:
  1. Memorize [2.37] on page 35 of the book
  2. Note that in the text we are given:

$$F(x) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \quad \text{where} \quad c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) f(x)$$

In our case this means

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\, dx$$

Now $c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) A x (a-x)\, dx$

$$c_n = A\sqrt{\frac{2}{a}} \left[a\int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx - \int_0^a x^2 \sin\left(\frac{n\pi x}{a}\right) dx \right.$$

Fortunately these integrals were given along with the text:

$$a\int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx = a\left[\frac{\sin(n\pi)\, a^2}{n^2\pi^2} - \frac{a^2 \cos(n\pi)}{n\pi} - \frac{\sin(0)}{n^2\pi^2} + \frac{0 \cdot \cos(0)}{n\pi}\right]$$

$$= \boxed{\frac{-a^3 \cos(n\pi)}{n\pi}}$$

$$-\int_0^a x^2 \sin\left(\frac{n\pi x}{a}\right) dx = \left[\frac{2a^3 \sin(n\pi)}{n^2\pi^2} + \left(\frac{a^3 2}{n^3\pi^3} - \frac{a^3}{n\pi}\right)\cos(n\pi) - 0 - \left(\frac{a^3 2}{n^3\pi^3} - 0\right)\cos(0)\right]$$

$$= \boxed{\frac{a^3}{n\pi}\cos(n\pi) - \frac{2a^3}{n^3\pi^3}\cos(n\pi) + \frac{2a^3}{n^3\pi^3}}$$

Combining the results: $\frac{2a^3}{n^3\pi^3}\left(\cos(n\pi) - 1\right)$

Now $c_n = A\sqrt{\frac{2}{a}} \frac{2a^3}{n^3\pi^3}\left(\cos(n\pi)-1\right) = \begin{cases} 0 & \text{for } n \text{ even} \\ -A\sqrt{\frac{2}{a}} \frac{4a^3}{n^3\pi^3} & \text{for } n \text{ odd} \end{cases}$

This can be further reduced by remembering that $|\Psi(x,0)|^2$ (as with all wave functions) represents a probability distribution & must integrate to 1.

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = \int_0^a |A|^2 x^2 (a-x)^2 dx$$

Expanding the inside $\rightarrow x^2(a-x)(a-x)$

$$x^2(a^2 - 2ax + x^2) = x^4 - 2ax^3 + a^2x^2$$

Now $1 = |A|^2 \int_0^a x^4 - 2ax^3 + a^2x^2 \, dx = |A|^2 \left[\left(\frac{1}{5}a^5 - 0\right) - \left(\frac{2a}{4}a^4 - 0\right) + \left(\frac{a^2}{3}a^3 - 0\right)\right]$

$$1 = |A|^2\left(a^5\left(\overset{6/30}{\frac{1}{5}} - \overset{15/30}{\frac{1}{2}} + \overset{10/30}{\frac{1}{3}}\right)\right) = |A|^2 \frac{a^5}{30} \quad \therefore \quad A = \sqrt{\frac{30}{a^5}}$$

Finally c_n for n odd $= -\sqrt{\frac{30}{a^5}}\sqrt{\frac{2}{a}} \frac{4a^3}{n^3\pi^3} = \boxed{\frac{8\sqrt{15}}{(n\pi)^3}}$

c) Remember that "the probability that a measurement of the energy would yield the value E_n" is $|c_n|^2$.

$$|c_n|^2 = \left|\frac{8\sqrt{15}}{(n\pi)^3}\right|^2 = \frac{64 \cdot 15}{(n\pi)^6} = \frac{960}{n^6\pi^6}$$

d) Remember that "the expectation value of the energy" can be written:

$$\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

This is because:

$$H\chi_n = E_n \chi_n \quad \text{where the } \chi_n\text{'s are stationary states}$$

$$\text{so } \langle H \rangle = \int \Psi^* H \Psi = \int \left(\sum c_m \chi_m\right)^* H \left(\sum c_n \chi_n\right) dx$$

$$= \sum \sum c_m^* c_n E_n \int \chi_m^* \chi_n \, dx = \sum |c_n|^2 E_n$$

$$\text{so, } \langle H \rangle = \sum_{n=1}^{\infty} \frac{960}{n^6\pi^6} E_n = \sum_{n=1}^{\infty} \frac{960}{n^6\pi^6} \frac{\hbar^2 n^2 \pi^2}{2ma^2} \quad \text{(see ✱✱)}$$

$$= \frac{960\hbar^2}{2\pi^4 ma^2} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

(thankfully this is given in the text as well).

$$= \frac{960\hbar^2}{2\pi^4 ma^2}\left(\frac{\pi^4}{96}\right) = \boxed{\frac{5\hbar^2}{ma^2}}$$

e) $$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{in^2\pi^2\hbar}{2ma^2}t}$$

$$= \sqrt{\frac{30}{a}} \left(\frac{2}{\pi}\right)^3 \sum_{n=\text{odd}}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{in^2\pi^2\hbar}{2ma^2}t}$$