Physics 451- Fall 2012

Homework # 7
Due Thursday, Sep 20, by 5pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
We will have help sessions twice a week:
Tuesday session (with Jacob): from 2 pm to 5pm – room 363 MARB
Thursday session (with Peter): from 3pm to 6pm – room 393 CB

List of problems (from the textbook):

2.19
2.20
2.21
2.22

Hints:
Useful integrals:

$$\int_{-\infty}^{\infty} e^{-\alpha u^2}\, du = \sqrt{\frac{\pi}{\alpha}}$$

$$\int_{-\infty}^{+\infty} f(|x|)dx = \int_{-\infty}^{0} f(-x)dx + \int_{0}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$$

grading: Total 40

2.19 — 5

2.20 — 10
a) 3
b) 2
c) 3
d) 2

2.21 — 10
a) 2
b) 3
c) 2
d) 3

2.22 — 15
a) 2
b) 4 (2 for ϕ, 2 for $\Psi(x,t)$)
c) 3
d) 4
e) 2

Homework # 7

Pb 2.19 Probability current J

$$J = \frac{i\hbar}{2m}\left(\Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x}\right)$$

Free particle $\Psi(x,t) = A\, e^{i\left(kx - \frac{\hbar k^2}{2m}t\right)}$

$$J(x,t) = \frac{i\hbar}{2m}|A|^2\left(e^{+i(kx-\omega t)}(-ik)\,e^{-i(kx-\omega t)} - e^{-i(kx-\omega t)}(ik)\,e^{i(kx-\omega t)}\right)$$

$$= \frac{i\hbar}{2m}|A|^2\,(-2ik) = \frac{2\hbar k|A|^2}{2m}$$

$J(x,t) = \frac{\hbar k}{m}|A|^2$ is constant and positive toward $(+x)$

↳ in space (for a given k)

$\boxed{\text{Pb } 2.20}$ Plancherel's theorem

a) Fourier series:

$$f(x) = \sum_{n=0}^{\infty}\left[a_n \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n\pi x}{a}\right)\right]$$

$$= \sum_{n=0}^{\infty}\left(a_n \frac{e^{\frac{in\pi x}{a}} - e^{-\frac{in\pi x}{a}}}{2i} + b_n \frac{e^{\frac{in\pi x}{a}} + e^{-\frac{in\pi x}{a}}}{2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(b_n - ia_n)}{2} e^{\frac{in\pi x}{a}} + \left(\frac{b_n + ia_n}{2}\right) e^{-\frac{in\pi x}{a}}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{\frac{in\pi x}{a}} \quad \text{with} \quad \boxed{\begin{cases} c_n = \dfrac{(b_n - ia_n)}{2} & \text{for } n > 0 \\ c_n = \dfrac{(b_{(-n)} + ia_{(-n)})}{2} & \text{for } n < 0 \end{cases}}$$

b) $$f(x)e^{-\frac{im\pi x}{a}} = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{in\pi x}{a}} e^{-\frac{im\pi x}{a}} = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{i(n-m)\pi x}{a}}$$

$$\int_{-a}^{+\infty} f(x) e^{-\frac{im\pi x}{a}} dx = \sum c_n \int_{-a}^{a} e^{\frac{i(n-m)\pi x}{a}} dx$$

$$= \sum c_n \, 2a \frac{\sin\left((n-m)\pi \frac{a}{a}\right)}{(n-m)\pi} = \delta_{nm} \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

$$= c_n \times 2a$$

So $$\boxed{c_n = \frac{1}{2a} \int_{-a}^{a} f(x) e^{-\frac{im\pi x}{a}} dx}$$

c) with $k = \frac{n\pi}{a}$ and $F(k) = \sqrt{\frac{2}{\pi}}\, a c_n$

Now: $f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x/a} = \sum_{n=-\infty}^{+\infty} \sqrt{\frac{\pi}{2}} \frac{F(k)}{a} e^{ikx}$

but $\boxed{\Delta k = \frac{\pi}{a}}$ so $\sqrt{\frac{\pi}{2}} \frac{1}{a} = \frac{1}{\sqrt{2\pi}} \Delta k$

So
$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} F(k) e^{ikx} \Delta k$$

And from (b): $c_n = \frac{1}{2a} \int_{-a}^{+a} f(x) e^{-in\pi x/a} dx$

So $F(k) = \sqrt{\frac{2}{\pi}}\, a c_n = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx$

d) If $a \longrightarrow \infty$: $\Delta k \to 0$ the <u>summation</u> becomes an <u>integration</u>

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} F(k) e^{ikx} dk$$

$\boxed{\text{Pb } 2.21}$ Free particle with initial wave function

$$\Psi(x,0) = A e^{-a|x|}$$

a) Normalization : $\displaystyle\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 = A^2 2\int_0^{\infty} e^{-2ax} dx = \frac{2A^2}{2a} = 1$

$$\Rightarrow \boxed{A = \sqrt{a}}$$

b) $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0) e^{-ikx} dx$$

$$= \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{+\infty} e^{-a|x|} e^{ikx} dx$$

$$= \sqrt{\frac{a}{2\pi}} \left(\int_0^{\infty} e^{(-a+ik)x} dx + \int_{-\infty}^{0} e^{(a+ik)x} dx \right)$$

$$= \sqrt{\frac{a}{2\pi}} \left(\int_0^{\infty} e^{(ik-a)x} dx + \int_0^{\infty} e^{-(a+ik)x} dx \right)$$

$$= \sqrt{\frac{a}{2\pi}} \int_0^{\infty} \left(e^{(ik-a)x} + e^{-(a+ik)x} \right) dx$$

$$= \sqrt{\frac{a}{2\pi}} \left[\frac{e^{(ik-a)x}}{ik-a} - \frac{e^{-(a+ik)x}}{a+ik} \right]_0^{\infty}$$

$$= \sqrt{\frac{a}{2\pi}} \left[\frac{-1}{ik-a} + \frac{1}{a+ik} \right] = \sqrt{\frac{a}{2\pi}} \left(\frac{a-ik+a+ik}{a^2+k^2} \right)$$

$$\boxed{\phi(k) = \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2}}$$

Pb. 2.21 (continued)

c) Wave function

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$$

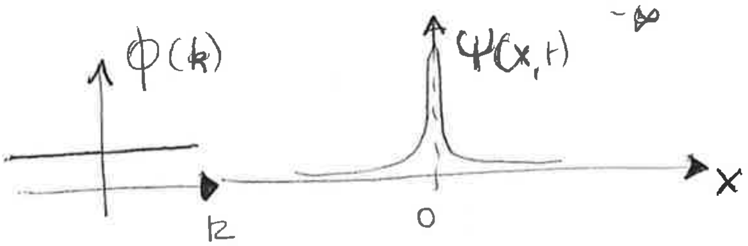
$$\Psi(x,t) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{+\infty} \frac{2a}{a^2+k^2}\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$$

d) • If $\underline{a \gg k}$ (a very large)

then $\dfrac{a\sqrt{a}}{a^2+k^2} \longrightarrow \dfrac{a^{3/2}}{a^2} = \dfrac{1}{\sqrt{a}}$ $\phi(k)$ is almost constant in k

$\phi(k) \simeq \dfrac{1}{\sqrt{2\pi a}}$

then $\Psi(x,t) \simeq \dfrac{1}{\sqrt{2\pi a}} \displaystyle\int_{-\infty}^{+\infty} e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$ is very sharp and narrow centered around $x = 0$

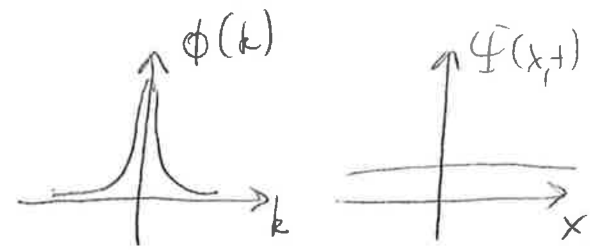


• If $\underline{a \ll k}$ (a very small)

then $\dfrac{a^{3/2}}{a^2+k^2} \simeq \dfrac{a^{3/2}}{k^2}$ $\phi(k)$ is very narrow at $k=0$

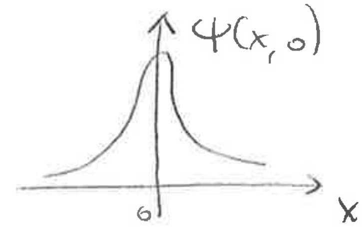
and $\Psi(x,t) \simeq \dfrac{a^{3/2}}{\pi} \displaystyle\int_{-\infty}^{+\infty} \frac{1}{k^2}\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$

Ψ is actually very flat in x (broad)



$\boxed{\text{Pb 2.22}}$ **The Gaussian Wave packet**

$$\Psi(x,0) = A\, e^{-ax^2} \quad (\text{Gaussian})$$



a) Normalization:

$$\int_{-\infty}^{+\infty} |\Psi(x,0)|^2 dx = A^2 \int_{-\infty}^{+\infty} e^{-2ax^2} dx = A^2 \sqrt{\frac{\pi}{2a}} = 1$$

$$\Rightarrow \quad A = \left(\frac{2a}{\pi}\right)^{1/4}$$

b) $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0)\, e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ikx} dx$$

change of variables: $y = \sqrt{a}\left(x + \frac{ik}{2a}\right)$ $\qquad dy = \sqrt{a}\, dx$

$$y^2 = a\left(x^2 - \frac{k^2}{4a^2} + \frac{ikx}{a}\right) = ax^2 + ikx - \frac{k^2}{4a}$$

So $$\phi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-y^2} e^{-\frac{k^2}{4a}} dy \frac{1}{\sqrt{a}}$$

$$= \frac{1}{\sqrt{2\pi a}} \left(\frac{2a}{\pi}\right)^{1/4} e^{-\frac{k^2}{4a}} \int_{-\infty}^{+\infty} e^{-y^2} dy = \frac{1}{\sqrt{2\pi a}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\pi}\, e^{-\frac{k^2}{4a}}$$

$$\boxed{\phi(k) = \frac{1}{\sqrt{2a}} \left(\frac{2a}{\pi}\right)^{1/4} e^{-\frac{k^2}{4a}}}$$ has also a Gaussian shape

Pb 2.22 (continued)

Now $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{(2a\pi)^{1/4}} \int_{-\infty}^{+\infty} e^{ikx}\, e^{-k^2\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)}\, dk$$

Change of variable:

$$y = \sqrt{\frac{1}{4a} + \frac{\hbar t}{2m}} \left(k - \frac{ix}{2\left(\frac{1}{4a} + \frac{i\hbar t}{2m}\right)} \right)$$

$$= \frac{1}{2\sqrt{a}} \left(1 + \frac{2\hbar t a}{m}\right)^{1/2} \left(k - \frac{ix}{\left(\frac{1}{2a} + \frac{i\hbar t}{m}\right)} \right)$$

$$y^2 = \left(\frac{1}{4a} + \frac{\hbar t}{2m}\right) \left(k^2 - \frac{2ikx}{\frac{1}{2a} + \frac{i\hbar t}{m}} + \frac{x^2}{\left(\frac{1}{2a} + \frac{i\hbar t}{m}\right)^2} \right)$$

$$= \left(\frac{1}{4a} + \frac{\hbar t}{2m}\right) k^2 - ikx - \frac{x^2}{2} \frac{2a}{1 + \frac{2i\hbar a t}{m}}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2a\pi)^{1/4}} \int_{-\infty}^{+\infty} e^{-y^2}\, e^{-ax^2 \frac{1}{1 + \frac{2i\hbar a t}{m}}}\, dy\, \frac{1}{\sqrt{\frac{1}{4a} + \frac{i\hbar t}{2m}}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sqrt{a}}{(2a\pi)^{1/4}}\, e^{-\frac{ax^2}{1 + \frac{2i\hbar a t}{m}}} \underbrace{\int_{-\infty}^{+\infty} e^{-y^2} dy}_{\sqrt{\pi}} \frac{1}{\left(1 + \frac{2i\hbar a t}{m}\right)^{1/2}}$$

$$\boxed{\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-\left[ax^2 / \left(1 + \frac{2i\hbar a t}{m}\right)\right]}}{\left(1 + \frac{2i\hbar a t}{m}\right)^{1/2}}}$$

c) $$|\Psi(x,t)|^2 = \left(\frac{2a}{\pi}\right)^{1/2} \left|\frac{e^{-ax^2/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}}\right|^2$$

$$= \left(\frac{2a}{\pi}\right)^{1/2} \frac{e^{-ax^2\left(\frac{1}{1+2i\hbar at/m}+\frac{1}{1-2i\hbar at/m}\right)}}{\sqrt{1+4\hbar^2a^2t^2/m^2}}$$

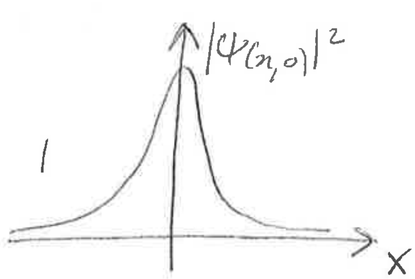
$$= \left(\frac{2a}{\pi}\right)^{1/2} \frac{e^{-ax^2\left(\frac{2}{1+4\hbar^2a^2t^2/m^2}\right)}}{\left(1+(2\hbar at/m)^2\right)^{1/2}}$$

using $\omega = \sqrt{\frac{a}{1+(2\hbar at/m)^2}}$ (depends on time)

$$\Rightarrow \boxed{|\Psi(x,t)|^2 = \left(\frac{2}{\pi}\right)^{1/2} e^{-2\omega^2x^2} \times \omega}$$

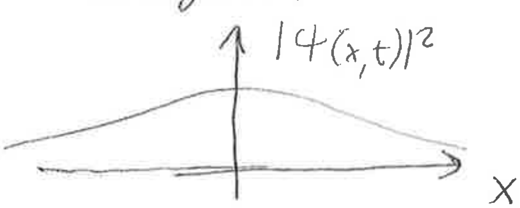
• At $t=0$ $\omega = \sqrt{a}$. $|\Psi(x,0)|^2 = \left(\frac{2}{\pi}\right)^{1/2} \sqrt{a}\, e^{-2a^2x^2}$

$|\Psi(x,0)|^2 = \left(\frac{2a}{\pi}\right)^{1/2} e^{-2a^2x^2}$ accordingly to (a)



$|\Psi(x,0)|^2$ has a gaussian shape

• At large t, $\omega \simeq \left(\frac{m}{2\hbar t}\right)\frac{1}{\sqrt{a}}$ $\Rightarrow$ $|\Psi(x,t)|^2 \propto \frac{1}{t} e^{-\frac{x^2}{t^2}}$



the shape broadens as time goes on

d) $\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^* x \Psi \, dx = \int_{-\infty}^{+\infty} x |\Psi|^2 dx$

$$= \left(\frac{2}{\pi}\right)^{1/2} \omega \underbrace{\int_{-\infty}^{+\infty} x e^{-2\omega^2 x^2} dx}_{0 \text{ (odd function)}} \qquad \boxed{\langle x \rangle = 0} \text{ at any time } t$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0 \qquad \boxed{\langle p \rangle = 0}$$

$$\langle x^2 \rangle = \left(\frac{2}{\pi}\right)^{1/2} \omega \int_{-\infty}^{+\infty} x^2 e^{-2\omega^2 x^2} dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \omega \frac{1}{4\omega^2} \sqrt{\frac{\pi}{2\omega^2}} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{4\omega^2}$$

$$\boxed{\langle x^2 \rangle = \frac{1}{4\omega^2}}$$

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(-\hbar^2 \frac{d^2}{dx^2}\right) \Psi \, dx$$

$\Psi = B e^{-\alpha x^2}$ with $B = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\theta}}$ and $\alpha = \frac{a}{1+i\theta}$

$$\frac{d\psi}{dx} = B(-2\alpha x) e^{-\alpha x^2}$$

$$\frac{d^2\psi}{dx^2} = B(-2\alpha + 4\alpha^2 x^2) e^{-\alpha x^2} \qquad \alpha + \alpha^* = \frac{2a}{1+\theta^2} = 2\omega^2$$

$$\langle p^2 \rangle = -\hbar^2 |B|^2 \int_{-\infty}^{+\infty} 2\alpha (2\alpha x^2 - 1) e^{-(\alpha+\alpha^*)x} dx$$

$$= -2\alpha \hbar^2 |B|^2 \left(2\alpha \frac{1}{4\omega^2} \sqrt{\frac{\pi}{2\omega^2}} - \sqrt{\frac{\pi}{2\omega^2}}\right) = -2\alpha \hbar^2 |B|^2 \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \left(\frac{\alpha}{2\omega^2} - 1\right)$$

$$\langle p^2 \rangle = -\hbar^2 \left(\frac{2a}{\pi}\right)^{1/2} \frac{1}{\sqrt{1+\theta^2}} \frac{2a}{1+i\theta} \sqrt{\frac{\pi}{2}} \frac{1}{\omega} \left(\frac{a}{1+i\theta} \frac{1}{2\omega^2} - 1\right) =$$

$$= -\hbar^2 \frac{a^{3/2}}{1+i\theta} \frac{2}{\omega} \frac{1}{\sqrt{1+\theta^2}} \left(\frac{a}{1+i\theta} \frac{1+\theta^2}{2a} - 1\right) = -\frac{2\hbar^2 a^{3/2}}{1+i\theta} \frac{1}{\sqrt{1+\theta^2}} \left(\frac{1-i\theta}{2} - 1\right) \frac{1}{\omega}$$

$$\langle p^2\rangle = \frac{2\hbar^2 a^{3/2}}{1+i\theta}\,\frac{1}{\sqrt{1+\theta^2}}\,\frac{1+i\theta}{2}\,\frac{1}{\omega} = \frac{\hbar^2 a^{3/2}}{\sqrt{1+\theta^2}\,\omega}$$

$$= \frac{\hbar^2 a^{3/2}}{\sqrt{1+\theta^2}}\,\frac{\sqrt{1+\theta^2}}{\sqrt{a}} = \frac{\hbar^2 a^{3/2}}{a^{1/2}}$$

$\boxed{\langle p^2\rangle = \hbar^2 a}$ (finally!)

e) $\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle = \frac{1}{4w^2} \Rightarrow \boxed{\sigma_x = \frac{1}{2w}}$

$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \langle p^2\rangle = \hbar^2 a \Rightarrow \boxed{\sigma_p = \hbar\sqrt{a}}$

$$\sigma_x \sigma_p = \frac{\hbar\sqrt{a}}{2\omega} = \frac{\hbar}{2}\sqrt{1+\left(\frac{2\hbar a t}{m}\right)^2}$$

since $1+\left(\frac{2\hbar a t}{m}\right)^2 \geq 1$ for any time t

$$\boxed{\sigma_x\sigma_p \geq \frac{\hbar}{2}}$$

The system comes the closest to uncertainty limit at $t=0$: $\sigma_x\sigma_p = \frac{\hbar}{2}$

when $|\Psi(x,0)|^2 = \left(\frac{2a}{\pi}\right)^{1/2} e^{-2a^2x^2}$ has a Gaussian shape