Physics 451- Fall 2012

Homework # 5

Due Friday, Sep 14, by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
We will have help sessions twice a week, in N337 ESC (undergraduate lab):
Tuesday session (with Jacob): from 2 pm to 5pm – room 363 MARB
Thursday session (with Peter): from 3pm to 6pm – room 393 CB

List of problems (from the textbook):

2.4
2.5
2.7
2.8

Hints:
Useful integrals for problem 2.4:

$$\int \sin^2(\alpha x)dx = \left[\frac{x}{2} - \frac{\sin(2\alpha x)}{4\alpha}\right]$$

$$\int x\sin^2(\alpha x)dx = \left[\frac{x^2}{4} - \frac{x\sin(2\alpha x)}{4\alpha} - \frac{\cos(2\alpha x)}{8\alpha^2}\right]$$

$$\int x^2\sin^2(\alpha x)dx = \left[\frac{x^3}{6} - \frac{x\cos(2\alpha x)}{4\alpha^2} - \left(\frac{x^2}{4\alpha} - \frac{1}{8\alpha^3}\right)\sin(2\alpha x)\right]$$

Useful integral for problem 2.5

$$\int x\sin(\alpha x)\sin(\beta x)dx = \frac{1}{2}\left[\frac{\cos\left[(\alpha-\beta)x\right]}{(\alpha-\beta)^2} - \frac{\cos\left[(\alpha+\beta)x\right]}{(\alpha+\beta)^2} + x\frac{\sin\left[(\alpha-\beta)x\right]}{(\alpha-\beta)} - x\frac{\sin\left[(\alpha+\beta)x\right]}{(\alpha+\beta)}\right]$$

Useful integrals for problem 2.7

$$\int x\sin(\alpha x)dx = \left[\frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha}\right]$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Grading guideline:

total 50

2.4 (10)	2.5 (20)	2.7 (15)	2.8 (5)
2 pts for $\langle x\rangle$	a) 2	a) 2	a) 2
2 " $\langle x^2\rangle$	b) 3	b) 5	b) 3
2 " $\langle p\rangle$	c) 5	c) 3	
2 $\langle p^2\rangle$	d) 5	d) 5	
2 $\sigma_x \sigma_p$	e) 5		

Homework #5

Pb 2.4 Particle in n^{th} stationary state

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right), \quad \text{energy } E_n = \frac{n^2\pi^2\hbar^2}{2ma}$$

$$\langle x\rangle = \int_0^a \psi_n^* \, x \, \psi_n \, dx$$

$$= \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left(\left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2n\pi x}{a}\right)}{4m\pi/a} - \frac{\cos(2n\pi x/a)}{8(n\pi/a)^2}\right]_0^a\right.$$

$$= \frac{2}{a}\left(\frac{a^2}{4} - \cancel{\frac{a\sin(2n\pi)}{4n\pi/a}} - \underbrace{\frac{\cos(2n\pi) - \cos(0)}{8(n\pi/a)^2}}_{0}\right)$$

$$\boxed{\langle x\rangle = \frac{a}{2}}$$ (independant of n)

$$\langle x^2\rangle = \int_0^a \psi_n^* x^2 \psi_n \, dx = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\left[\frac{x^3}{6} - \frac{x}{4(n\pi/a)^2}\cos\left(\frac{2n\pi}{a}x\right) - \underbrace{\left(\frac{ax^2}{4n\pi} - \frac{1}{8(n\pi/a)^3}\right)\sin\left(\frac{2n\pi x}{a}\right)}_{0}\right]_0^a$$

$$\langle x^2\rangle = \frac{2}{a}\left(\frac{a^3}{6} - \frac{a^3}{4n^2\pi^2}\cos(2n\pi)\right)$$

$$\boxed{\langle x^2\rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)}$$

- $\langle p\rangle = m\dfrac{d\langle x\rangle}{dt} = 0$ since $\langle x\rangle$ does not depend on t

- $\langle p^2\rangle = \displaystyle\int_0^a \psi_n^*\left(-i\hbar\frac{\hat{\partial}}{\partial x}\right)^2\psi_n\,dx = \int_0^a \psi_n^*\,(-\hbar^2)\frac{d^2\psi_n}{dx^2}\,dx$

$$= -\hbar^2\int_0^a \frac{2}{a}\sin\left(\frac{n\pi x}{a}\right)\left(-\frac{n^2\pi^2}{a^2}\right)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2\hbar^2 n^2\pi^2}{a^3}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2n^2\pi^2\hbar^2}{a^3}\left[\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{a}\right)}{\frac{4m\pi}{a}}\right]_0^a$$

$$= \frac{2n^2\pi^2\hbar^2}{a^3}\left(\frac{a}{2}\right)$$

$$\boxed{\langle p^2\rangle = \left(\frac{m\pi\hbar}{a}\right)^2}$$

$\left(\text{By the way: } E_n = \dfrac{\langle p^2\rangle}{2m} = \dfrac{n^2\pi^2\hbar^2}{2ma^2}\right)$

<u>Finally</u>: • $\sigma_x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left[a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \frac{a^2}{4}\right]^{1/2}$

$$\sigma_x = a\left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)^{1/2}$$

- $\sigma_p = \left(\langle p^2\rangle - \langle p\rangle^2\right)^{1/2} = \langle p^2\rangle^{1/2} = \dfrac{n\pi\hbar}{a}$

Pb 2.4 (continued)

$$\sigma_x \sigma_p = \frac{n\pi\hbar}{a} \cdot a \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)^{1/2}$$

$$= n\pi\hbar \left(\frac{1}{12} - \frac{1}{2n^2\pi^2}\right)^{1/2}$$

$$= \frac{\hbar}{2} \left(\frac{n^2\pi^2}{3} - \frac{2\cancel{n^2\pi^2}}{\cancel{n^2\pi^2}}\right)^{1/2}$$

$$\boxed{\sigma_x \sigma_p = \frac{\hbar}{2} \left(\frac{(n\pi)^2}{3} - 2\right)^{1/2}}$$

Since $\frac{\pi^2}{3} > 2$ (in fact $\frac{\pi^2}{3} - 2 = 1.289\ldots$)

$\Rightarrow$ $\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$ for ANY value n

$\Rightarrow$ The particle satisfies the Heisenberg's uncertainty principle

The closest to the uncertainty limit is the "ground state" for $n = 1$

we have $\sigma_x \sigma_p = \frac{\hbar}{2} \left(\frac{\pi^2}{3} - 2\right)^{1/2} \simeq 1.135 \left(\frac{\hbar}{2}\right)$

$\boxed{\text{Pb } 2.5}$ A particle in infinite square well

$$\Psi(x,0) = A\left(\psi_1(x) + \psi_2(x)\right)$$

(a) Normalization : $\Psi(x,0)$ normalized at $t=0$, $\Psi(x,t)$ will be normalized at any time t

$$\int_0^a |\Psi|^2 dx = A^2 \int_0^a (\psi_1^* + \psi_2^*)(\psi_1+\psi_2)\,dx$$

$$= A^2\Big(\underbrace{\int_0^a \psi_1^*\psi_1}_{1} + \underbrace{\int_0^a \psi_2^*\psi_2}_{1} + \underbrace{\int_0^a \psi_1^*\psi_2}_{\delta_{12}=0} + \underbrace{\int_0^a \psi_1\psi_2^*}_{\delta_{21}=0}\Big)$$

$$= 2A^2 = 1 \qquad (\psi_n \text{ are orthonormal})$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{2}}}$$

(b) $$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left(\sin\left(\frac{\pi x}{a}\right)e^{-iE_1t/\hbar} + \sin\left(\frac{2\pi x}{a}\right)e^{-iE_2t/\hbar}\right)$$

$$\rho = |\Psi(x,t)|^2 = \frac{1}{\sqrt{a}^2}\left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\left(e^{-i(E_1-E_2)t/\hbar} + e^{-i(E_2-E_1)t/\hbar}\right)\right)$$

$$= \frac{1}{a}\left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right)$$

Pb 2.5 (continued)

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$\rho(x,t) = |\Psi(x,t)|^2$$

$$= \frac{1}{a}\left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{\pi^2 \hbar^2}{2ma^2\hbar}(4-1)t\right)\right)$$

$$\boxed{\rho(x,t) = \frac{1}{a}\left(\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right)}$$

(c) $$\langle x \rangle = \int_0^a \Psi^* x \Psi \, dx$$

$$= \frac{1}{2}\int_0^a \left(\psi_1^*(x) e^{+iE_1 t/\hbar} + \psi_2^* e^{iE_2 t/\hbar}\right) x \left(\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}\right) dx$$

$$= \frac{1}{2}\left[\int_0^a \psi_1^* x \psi_1 \, dx + \int_0^a \psi_2^* x \psi_2 \, dx + \int_0^a \psi_1^* x \psi_2 e^{i(E_1 - E_2)t/\hbar} dx + \int_0^a \psi_1 x \psi_2^* e^{-i\ldots} d\ldots\right.$$

$$= \frac{2}{2a}\left[\int_0^a x \sin^2\left(\frac{\pi x}{a}\right) dx + \int_0^a x \sin^2\left(\frac{2\pi x}{a}\right) dx + 2\cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)\int_0^a x \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi\ldots}{a}\right.\right.$$

$$= \frac{2}{2a}\left[\left[\frac{x^2}{4} - \frac{x \sin\left(\frac{2\pi x}{a}\right)}{\frac{4\pi}{a}} - \frac{\cos\left(\frac{2\pi x}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right]_0^a + \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{4\pi x}{a}\right)}{8\pi/a} - \frac{\cos\left(\frac{4\pi x}{a}\right)}{8\left(\frac{2\pi}{a}\right)^2}\right]_0^a\right.$$

$$+ \frac{2\cos}{2}\left(\frac{(E_2 - E_1)t}{\hbar}\right)\left[\frac{\cos\left(\frac{\pi x}{a}\right)}{\left(\frac{\pi}{a}\right)^2} - \frac{\cos\left(\frac{3\pi}{a}x\right)}{\left(\frac{3\pi}{a}\right)^2} + x\frac{\sin\left(\frac{\pi x}{a}\right)}{\left(\frac{\pi}{a}\right)} - x\frac{\sin\left(\frac{3\pi x}{a}\right)}{\left(\frac{3\pi}{a}\right)}\right]_0^a$$

$$\langle x \rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} + \cos(3\omega t)\left(\frac{-2}{\left(\frac{\pi}{a}\right)^2} + \frac{2}{\left(\frac{3\pi}{a}\right)^2}\right)\right]$$

$$= \frac{1}{a}\left[\frac{a^2}{2} + 2\cos(3\omega t)\,\frac{a^2}{\pi^2}\left(-1+\frac{1}{9}\right)\right]$$

$$= a\left[\frac{1}{2} + \frac{2}{\pi^2}\left(-\frac{8}{9}\right)\cos(3\omega t)\right]$$

$$\boxed{\langle x \rangle = \frac{a}{2}\left(1 - \frac{32}{9\pi^2}\cos(3\omega t)\right)}$$

$\langle x \rangle$ is a function of time:
It oscillates in time with a frequency 3ω
and an oscillating amplitude of $\frac{16a}{9\pi^2}$

d) Again we defined $\langle p \rangle = m\frac{d\langle x \rangle}{dt}$

So we can just use the calculated $\langle x \rangle$

$$\langle p \rangle = \frac{ma}{2} \times \frac{32}{9\pi^2} \cdot 3\omega \sin(3\omega t)$$

$$\boxed{\langle p \rangle = \frac{32}{6\pi^2}\, m a \omega \sin(3\omega t)}$$

with $\omega = \frac{\pi^2 \hbar}{2ma^2}$ $\Rightarrow$ $\boxed{\langle p \rangle = \frac{8}{3}\frac{\hbar}{a}\sin(3\omega t)}$

d) Measuring the energy of $\Psi(x,t)$

$$E = \langle \hat{H} \rangle$$

$$= \int_0^a \bar{\Psi}\, \hat{H}\, \Psi \, dx$$

$$= \frac{1}{2}\int_0^a \left(\psi_1^* e^{+\frac{iE_1t}{\hbar}} + \psi_2^* e^{\frac{iE_2t}{\hbar}}\right) \hat{H} \left(\psi_1 e^{-\frac{iE_1t}{\hbar}} + \psi_2 e^{-\frac{iE_2t}{\hbar}}\right) dx$$

$$\hat{H}\psi_1 = E_1\psi_1 \quad ; \quad \hat{H}\psi_2 = E_2\psi_2$$

$$\Rightarrow E = \frac{1}{2}\int_0^a \left(\psi_1^* e^{\frac{iE_1t}{\hbar}} + \psi_2^* e^{\frac{iE_2t}{\hbar}}\right)\left(E_1\psi_1 e^{-\frac{iE_1t}{\hbar}} + E_2\psi_2 e^{-\frac{iE_2t}{\hbar}}\right) dx$$

$$= \frac{1}{2}\left(E_1 \underbrace{\int_0^a |\psi_1|^2 dx}_{1} + E_2 \underbrace{\int_0^a |\psi_2|^2 dx}_{1} + E_1E_2 \underbrace{\int \psi_1\psi_2\, dx}_{0}\, 2\cos 3\omega t\right)$$

$$\boxed{E = \frac{1}{2}\left(E_1 + E_2\right)}$$

in other terms: we get E_1 or E_2 with equal probabilities

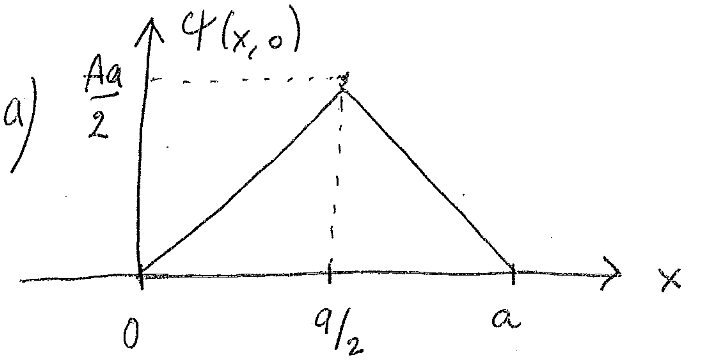
$$\text{So } E = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2ma^2}\right)(1+4) = \boxed{\frac{5}{2}\frac{\pi^2\hbar^2}{ma^2} = E}$$

E is the "average value" of E_1 and E_2

3/11

$\boxed{\text{Pb 2.7}}$ A particle in the initial state:

$$\Psi(x,0) = \begin{cases} A x & 0 \le x \le \frac{a}{2} \\ A(a-x) & \frac{a}{2} \le x \le a \end{cases}$$

a) 

Normalization:
$$\int_{-\infty}^{+\infty} |\Psi|^2 dx = \int_0^{a/2} A^2 x^2 dx + \int_{a/2}^{a} A^2 (a-x)^2 dx$$

$$= A^2 \left(\frac{x^3}{3} \Big]_0^{a/2} - \frac{(a-x)^3}{3} \Big]_{a/2}^{a} \right)$$

$$= A^2 \left(\frac{a^3}{3\times 8} - \left(- \frac{(a/2)^3}{3} \right) \right)$$

$$1 = \frac{2}{3\times 8} a^3 A^2$$

$$\boxed{A = \sqrt{\frac{12}{a^3}}}$$

b) $$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n e^{-i\frac{E_n t}{\hbar}}$$

To find c_n: Dirichlet's theorem $c_n = \int_{-\infty}^{+\infty} \psi_n^* \Psi(x,0)\, dx$

Pb 2.7 (continued)

$$c_n = \int_{-\infty}^{+\infty} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\, dx$$

$$= \int_0^{a/2} \sqrt{\frac{2}{a}} A x \sin\left(\frac{n\pi x}{a}\right) dx + \int_{a/2}^{a} \sqrt{\frac{2}{a}} A (a-x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2\sqrt{6}}{a^2}\left[\frac{\sin\left(\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2} - \frac{x\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right]_0^{a/2} + \frac{2\sqrt{6}}{a^2}\,\frac{a^2}{n\pi}\left[-\cos\frac{n\pi x}{a}\right]_{a/2}^{a}$$

$$- \frac{2\sqrt{6}}{a^2}\left[\frac{\sin\left(\frac{n\pi x}{a}\right)}{\left(\frac{n\pi}{a}\right)^2} - \frac{x\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right]_{a/2}^{a}$$

$$= \frac{2\sqrt{6}}{a^2}\left[\cancel{-\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{a^2}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)} - \frac{\sin(n\pi)}{\left(\frac{n\pi}{a}\right)^2} + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{a}\right)^2}\right.$$

$$\left. + \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{a}\right)^2} + \cancel{\frac{a\cos(n\pi)}{\frac{n\pi}{a}}} - \cancel{\frac{\frac{a}{2}\cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{a}}}\right]$$

$$= \frac{2\sqrt{6}}{a^2}\,\frac{a^2}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \underbrace{\cancel{\sin(n\pi)}}_{0}\right)$$

$$\boxed{c_n = \frac{4\sqrt{6}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)} = \begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^{\frac{n-1}{2}}\frac{4\sqrt{6}}{n^2\pi^2} & \text{for } n \text{ odd} \end{cases}$$

Pb 2.7 (continued)

c) Probability of measuring E_1 is given by the coefficient in front of E_1 in $\langle \hat{H} \rangle$:

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

for E_1: probability $c_1^2 = \left(\frac{-4\sqrt{6}}{\pi^2} \right)^2 = \frac{16 \times 6}{\pi^4} = 0.9855\ldots$

$\sim \underline{98.5\%}$

d) Expectation value of the energy:

$$\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

$$= \sum_{n=0}^{\infty} |c_{2n+1}|^2 E_{2n+1}$$

$$= \sum_{n=0}^{\infty} \left((-1)^{2n} \frac{4\sqrt{6}}{(2n+1)^2 \pi^2} \right)^2 \frac{(2n+1)^2 \pi^2 \hbar^2}{2ma^2}$$

$$= \sum_{n=0}^{\infty} (-1)^{2n} \frac{(4\sqrt{6})^2}{(2n+1)^2 \pi^4} \frac{\pi^2 \hbar^2}{2ma^2} = \sum_{n=0}^{\infty} \frac{16 \times 6}{(2n+1)^2} \frac{\hbar^2}{2ma^2\pi^2}$$

$$= \frac{16 \times 6\, \hbar^2}{2ma^2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\boxed{\langle \hat{H} \rangle = \frac{6\hbar^2}{ma^2}}$$