Physics 451- Fall 2012

Homework #22
Due Thursday, Nov 29, by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
Help sessions twice a week: from 4pm to 6pm – room 337 ESC

List of problems (from the textbook):

5.1
5.2
5.4
5.6

Hints: Problem **5.1**: a) The old coordinates are $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$. The new coordinates are $\vec{R} = (X, Y, Z)$ and $\vec{r} = (x, y, z)$. To express the "Del" operator in new coordinates, find an expression for each of the components along x,y,z, and use partial derivatives. For example, use: $\nabla_{1,x} = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x}$

b) Express the Hamiltonian in terms of old coordinates first and transform its expression in terms in new coordinates.
c) Divide the Schrödinger equation by $\psi_r \psi_R$ and separate a term that depends on r only from a term that depends on R only.

Problem **5.2**: numerical application

a) Hydrogen atom (1 proton, 1 electron): $m_p = 1.67\times10^{-27}\,kg$; $m_e = 9.1\times10^{-31}\,kg$
b) Deuterium atom (1 proton + 1neutron, 1 electron): $m_n = m_p$ so $m_{nucleus} = 2m_p$
c) Positronium atom (1 positron, 1 electron): $m_{pos} = m_e$
d) Muonic hydrogen (1 proton, 1 muon): $m_{muon} = 206.77 m_e$

Problem **5.6:** the stationary states of the infinite square well are: $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

To find $\langle(\Delta x)^2\rangle$ in the three cases, you will basically need to calculate:

$\langle x\rangle_n$, $\langle x^2\rangle_n$ and $\langle x\rangle_{nl}$ (by integration – between limits 0 and a)

$$\int x^2 \sin^2(\alpha x)dx = \left[\frac{x^3}{6} - \frac{x\cos(2\alpha x)}{4\alpha^2} - \left(\frac{x^2}{4\alpha} - \frac{1}{8\alpha^3}\right)\sin(2\alpha x)\right]$$

$$\int x\sin(\alpha x)\sin(\beta x)dx = \frac{1}{2}\left[\frac{\cos[(\alpha-\beta)x]}{(\alpha-\beta)^2} - \frac{\cos[(\alpha+\beta)x]}{(\alpha+\beta)^2} + x\frac{\sin[(\alpha-\beta)x]}{(\alpha-\beta)} - x\frac{\sin[(\alpha+\beta)x]}{(\alpha+\beta)}\right]$$

grading

$\boxed{5.1}$ a) 4 b) 4 c) 2 — 10

$\boxed{5.2}$ a) 2 b) 4 c) 2 d) 2 — 10

$\boxed{5.4}$ a) 2 b) 2 — 4

$\boxed{5.6}$ a) 6 b) 6 c) 4 — 16

Homework 22

Pb 5.1 $\vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ (position of center of mass)

and $\vec{r} = \vec{r}_1 - \vec{r}_2$

a)

$\vec{r}_1 = \vec{r}_2 + \vec{r}$

$\vec{r}_2 = \vec{r}_1 - \vec{r}$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r})}{m_1 + m_2} = \frac{m_1 + m_2}{m_1 + m_2}\vec{r}_1 - \frac{m_2}{m_1 + m_2}\vec{r}$$

$$\Rightarrow \boxed{\vec{r}_1 = \vec{R} + \frac{\mu}{m_1}\vec{r}}$$

with $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

Similarly $\boxed{\vec{r}_2 = \vec{R} - \dfrac{\mu}{m_2}\vec{r}}$

$$\vec{\nabla}_1 = (\nabla_{1,x}, \nabla_{1,y}, \nabla_{1,z})$$

$\vec{R} = (X, Y, Z)$

$\vec{r} = (x, y, z)$

$$\nabla_{1x} = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x}$$

but $\begin{cases} \dfrac{\partial X}{\partial x_1} = \dfrac{m_1}{m_1 + m_2} \\ \dfrac{\partial x}{\partial x_1} = 1 \end{cases}$

$$\frac{\partial}{\partial x_1} = \frac{m_1}{m_1 + m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$$

same for every component $\Rightarrow$ $\boxed{\nabla_1 = \dfrac{\mu}{m_2}\nabla_R + \nabla_r}$

similarly $\boxed{\nabla_2 = \dfrac{\mu}{m_1}\nabla_R + \nabla_r}$

b) The hamiltonian for the 2 particles is:

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(r_1, r_2)$$

$$= -\frac{\hbar^2}{2m_1}\left(\left(\frac{\mu}{m_2}\right)^2\nabla_R^2 + \nabla_r^2 + \frac{2\mu}{m_2}\nabla_R\nabla_r\right) - \frac{\hbar^2}{2m_2}\left(\left(\frac{\mu}{m_1}\right)^2\nabla_R^2 + \nabla_r^2 - \frac{2\mu}{m_1}\nabla_R\nabla_r\right)$$

$$+ V\left(R + \frac{\mu}{m_1}r,\ R - \frac{\mu}{m_2}r\right)$$

$$= -\frac{\hbar^2}{2}\left(\frac{\mu^2}{m_1 m_2^2}\nabla_R^2 + \frac{\nabla_r^2}{m_1} + \frac{\mu^2}{m_2 m_1^2}\nabla_R^2 + \frac{\nabla_r^2}{m_2}\right) + V(R, r)$$

$$= -\frac{\hbar^2}{2}\left(\left(m_1 + m_2\right)\frac{1}{(m_1+m_2)^2}\nabla_R^2 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\nabla_r^2\right) + V(r)$$

$$= -\frac{\hbar^2}{2}\left(\frac{1}{m_1+m_2}\nabla_R^2 + \frac{1}{\mu}\nabla_r^2\right) + V(r)$$

$$= -\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)$$

Schroedinger equation : stationary states

$$\Rightarrow \boxed{H\psi = \left(-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right)\psi = E\psi}$$

c) <u>Separation of variables</u> $\psi(R, r) = \psi_R(R)\,\psi_r(r)$

Dividing the Schroedinger equation by $\psi_R \psi_r$:

$$\underbrace{\left(-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\right)}_{\text{function of } R \text{ only}} + \underbrace{\left(-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right)}_{\text{function of } r \text{ only}} = E \leftarrow \text{constant}$$

For this to be true, we have two independant equations:

$$\boxed{-\frac{\hbar^2}{2(m_1+m_2)}\nabla_R^2\psi_R = E_R\psi_R}$$

(function of R only)

E_R = energy of center of mass

and

$$\boxed{\left(-\frac{\hbar^2}{2\mu}\nabla_r^2 + V(r)\right)\psi_r = E_r\psi_r}$$

(function of r only)

E_r = energy of the 2 partids in the frame of the center of mass

The particle 2 motion in respect to particle 1 is the same than than if we have one single particle with reduced mass μ in potential V

Pb 5.2

Hydrogen atom

$m_N = 1.67\ 10^{-27}$ kg

$m_e = 9.1\ 10^{-31}$ kg.

a) $E_r = \frac{\mu}{m_e}E_1 = \frac{m_e m_N}{m_e(m_e+m_N)}E_1 = \frac{m_N}{m_e+m_N}E_1$

$$\frac{\Delta E}{E_1} = \frac{m_e}{m_e+m_N} = \frac{9.1\ 10^{-31}}{1.67\ 10^{-27}+9.1\ 10^{-31}} \simeq \frac{9.1\ 10^{-4}}{1.67} = 5.44\ 10^{-4} = 0.054\ \%$$

b) For hydrogen: $m_p + m_e$ $\qquad \mu = \frac{m_e m_p}{m_p+m_e}$

For deuterium: $2m_p + m_e$ $\qquad \mu' = \frac{m_e(2m_p)}{m_p+m_e}$

Rydberg formulae: $\frac{1}{\lambda} = R\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$ with R proportional to mass

$$\Rightarrow \frac{\Delta(1/\lambda)}{1/\lambda} = \frac{\Delta R}{R} = \frac{\Delta\mu}{\mu} \quad \text{and} \quad \frac{\Delta(1/\lambda)}{1/\lambda} = -\frac{\Delta\lambda}{\lambda}$$

$$\text{So}\quad \frac{\Delta\lambda}{\lambda} = -\frac{\Delta\mu}{\mu} = \frac{\mu' - \mu}{\mu} = \cancel{\frac{m_e}{m_e+m_p}\left(2m_p - m_p\right) - \frac{m_e m_p}{m_e+m_p}}$$

$$= m_e\left(\frac{2m_p}{m_e+2m_p} - \frac{m_p}{m_e+m_p}\right)$$

$$= m_e m_p\left(\frac{2m_e + 2m_p - m_e - 2m_p}{(m_e+2m_p)(m_e+m_p)}\right)$$

$$\boxed{\frac{\Delta\lambda}{\lambda} = m_e m_p \frac{m_e}{(m_e+2m_p)(m_e+m_p)} = \frac{m_e}{(m_e+2m_p)}\,\mu}$$

$$\frac{\Delta\lambda}{\lambda} \simeq \frac{9.1\ 10^{-31}}{9.1\ 10^{-31} + 2\times 1.67\ 10^{-27}} \simeq \frac{9.1}{2\times 1.67}\ 10^{-4} = 2.72\ 10^{-4} = 0.027\%$$

But:

$$\lambda = \frac{1}{R}\frac{1}{\left(\frac{1}{2^2} - \frac{1}{3^2}\right)} = \frac{36}{5R} = \underline{6.563\ 10^{-7}\ \text{m}}$$

$$\rightarrow \boxed{\Delta\lambda = 1.79\ 10^{-10}\ \text{m}}$$

c) Positronium:

$$\mu = \frac{m_e^2}{2m_e} = \frac{m_e}{2}$$

$$\Rightarrow \boxed{|E_1'| = \frac{|E_1|}{2} = 6.8\ \text{eV}}$$

d) Muonic hydrogen:

$$\mu = \frac{m_\mu m_p}{m_\mu + m_p} \quad \text{with } m_\mu = 206.77\, m_e$$

$$\lambda = \frac{1}{R_\mu}\frac{1}{1-\frac{1}{4}} = \frac{4}{3R_\mu} \quad \text{with } \boxed{R_\mu = \frac{m_\mu}{m_e}\frac{m_e+m_p}{m_\mu+m_p}R}$$

$$\lambda = \frac{4}{3}\frac{m_e}{m_\mu}\frac{m_\mu+m_p}{m_e+m_p}\frac{1}{R} \simeq \frac{4}{3}\frac{m_e}{m_\mu}\frac{1}{R} \simeq \underline{\underline{6.54\ .10^{-10}\ \text{m}}} \quad (\text{X rays})$$

Pb 5.4

a) ψ_a & ψ_b are orthonormal

$$\psi_\pm = A\left(\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\right)$$

$$\|\psi_\pm\| = |A|^2\left(\langle \psi_a^*\psi_b^* \pm \psi_b^*\psi_a^* \,|\, \psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\rangle\right)$$

$$= |A|^2\Big(\langle\psi_a^*|\psi_a\rangle\langle\psi_b^*|\psi_b\rangle + \langle\psi_b^*|\psi_b\rangle\langle\psi_a^*|\psi_a\rangle$$

$$\pm 2\,\underbrace{\langle\psi_b^*|\psi_a\rangle}_{0}\underbrace{\langle\psi_a^*|\psi_b\rangle}_{0}\Big)$$

$$= |A|^2(1 + 1) = 2|A|^2 = 1$$

$$\Rightarrow \boxed{|A| = 1/\sqrt{2}}$$

b) If $\psi_a = \psi_b$ (only for bosons)

$$\psi_+ = 2A(\psi_a(r_1)\psi_a(r_2)) = 2A\psi_a(r_1)\psi_a(r_2)$$

$$\|\psi_+\| = 4|A|^2\langle\psi_a^*|\psi_a\rangle^2 = 4|A|^2$$

$$\Rightarrow |A|^2 = 1/4 \qquad \boxed{|A| = \frac{1}{2}}$$

$\boxed{\text{Pb 5.6}}$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\psi_\ell(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\ell\pi x}{a}\right)$$

a) Distinguishable particles:

$$\langle (x_2 - x_1)^2 \rangle = \langle x^2 \rangle_n + \langle x^2 \rangle_\ell - 2 \langle x \rangle_n \langle x \rangle_\ell$$

$$\langle x \rangle_m = \frac{a}{2} = \langle x \rangle_\ell$$

$$\langle x^2 \rangle_n = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$$

$$\langle x^2 \rangle_\ell = a^2 \left(\frac{1}{3} - \frac{1}{2(\ell\pi)^2}\right)$$

$$\langle (\Delta x)^2 \rangle_d = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2} + \frac{1}{3} - \frac{1}{2(\ell\pi)^2}\right) - \frac{a^2}{2}$$

$$= a^2\left(\frac{2}{3} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{\ell^2}\right) - \frac{1}{2}\right) = \boxed{a^2\left(\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{\ell^2}\right)\right) = \langle (\Delta x)^2 \rangle_d}$$

b) Identical bosons.

$$\langle (\Delta x)^2 \rangle = \langle (\Delta x)^2 \rangle_d - 2\left|\langle x \rangle_{n\ell}\right|^2$$

$$\langle x \rangle_{n\ell} = \frac{2}{a}\int_0^a x \sin\left(\frac{\ell\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{1}{a}\left[\left(\frac{a}{(m-n)\pi}\right)^2\left(\cos(m-n)\pi - 1\right) - \left(\frac{a}{(m+n)\pi}\right)^2\left(\cos(m+n\,\pi) - 1\right)\right]$$

$$\cos(m-n)\pi = \cos(m+n)\pi = (-1)^{m+n}$$

$$\langle x \rangle_{n\ell} = \frac{1}{a}\left((-1)^{m+n} - 1\right)\frac{a^2}{\pi^2}\left(\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2}\right) = \begin{cases} \frac{a}{\pi^2}\frac{8mn}{(m^2-n^2)^2} & \text{for } m+n \text{ odd} \\ 0 & \text{for } m+n \text{ even} \end{cases}$$

Pb 5.6 (cont.)

So For identical Bosons:

$$\langle(\Delta x)^2\rangle = \langle(\Delta x)^2\rangle_d - 2\,|\langle x\rangle_{ne}|^2$$

$$\boxed{\langle(\Delta x)^2\rangle_{\text{Bosons}} = a^2\left(\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{m^2} + \frac{1}{p^2}\right)\right) - \frac{2a}{\pi^2}\,\frac{8mn}{(m^2-n^2)^2}}$$

for $(m+n)$ odd

If $(m+n)$ even, then $\langle(\Delta x)^2\rangle_{\text{Boson}} = \langle(\Delta x)^2_d\rangle$

) For identical fermions:

$$\langle(\Delta x)^2\rangle = \langle(\Delta x)^2\rangle_d + 2\,|\langle x\rangle_{ne}|^2$$

$$\boxed{\langle(\Delta x)^2\rangle_{\text{fermions}} = a^2\left(\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{\ell^2}\right)\right) + \frac{2a}{\pi^2}\,\frac{8mn}{(m^2-n^2)^2}}$$

for $(m+n)$ odd

If $(m+n)$ even, then $\langle(\Delta x)^2\rangle_{\text{Fermions}} = \langle(\Delta x)^2\rangle_{\text{Bosons}} = \langle(\Delta x)^2\rangle_d$