Physics 451- Fall 2012

Homework #20

Due Thursday, Nov 15, by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
Help sessions T Th 4 to 6pm N 337 ESC

List of problems (from the textbook):

4.18
4.19
4.21
4.22

Hints:

For problem **4.18**: The point where to start is to calculate the norm (or the norm square) of the vector $L_{\pm} f_l^m$ in order to determine the expression for A_l^m (or its square). By doing so, you will introduce the product $L_{\pm}L_{\mp}$ which you can replace by its expression in terms of L^2 and L_z (using equation 4.112). Use the fact that f_l^m are normalized.

For problem **4.21**, use a test function in order to apply all the successive derivations.
Trigonometric relationships: $\frac{d}{d\theta}\cot\theta = -\csc^2\theta = -\frac{1}{\sin^2\theta}$ and $\csc^2\theta = 1+\cot^2\theta$

Useful integral for problem **4.22**

$$\int_0^{\pi} \sin^{2l+1}\theta d\theta = 2\frac{2*4*6....(2l)}{1*3*5...(2l+1)} = 2\frac{(2^l l!)^2}{(2l+1)!}$$

For question c), remember the integral for angular normalization: $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} |Y_l^m|^2 \sin\theta d\theta d\phi$

grading : $\boxed{4.18}$ 5pts

$\boxed{4.19}$ a) 6 (1pt each)
b) 2 pts
c) 4 pts
d) 3 pts
(15)

$\boxed{4.21}$ a) 5 pts
b) 5 pts

$\boxed{4.22}$ a) 2 pts
b) 6 pts
c) 2 pts

Total
(40)

$\boxed{\text{Homework 20}}$

$\boxed{\text{Pb 4.18}}$ $$L_\pm f_\ell^m = A_\ell^m f_\ell^{m\pm 1}$$ Find A_ℓ^m

We assume that the eigenvectors f_ℓ^m are normalized.

Let's calculate $$\| L_\pm f_\ell^m \|^2 = |A_\ell^m|^2 \underbrace{\| f_\ell^m \|^2}_{1} = |A_\ell^m|^2$$

On the other hand: $$\| L_\pm f_\ell^m \|^2 = \langle L_\pm^\dagger f_\ell^m | L_\pm f_\ell^m \rangle$$

We need to calculate the Hermitian conjugate of $L_\pm$: $$= \langle f_\ell^m | L_\pm^\dagger L_\pm | f_\ell^m \rangle$$

$$L_\pm = L_x \pm i L_y$$
$$L_\pm^\dagger = L_x^\dagger \mp i L_y^\dagger$$

but L_x and L_y are observables soo they are hermitian $\left\} \begin{aligned} L_y^\dagger &= L_y \\ L_x^\dagger &= L_x \end{aligned} \right.$

$$\Rightarrow \quad L_\pm^\dagger = L_x \mp i L_y = L_\mp$$

$$\Rightarrow \quad \| L_\pm f_\ell^m \|^2 = \langle f_\ell^m | L_\mp L_\pm | f_\ell^m \rangle$$

From equation 4.112, we know that $L_\mp L_\pm = L^2 - L_z^2 \pm \hbar L_z$

$$\Rightarrow \quad |A_\ell^m|^2 = \langle f_\ell^m | (L^2 - L_z^2 \pm \hbar L_z) | f_\ell^m \rangle$$
$$= \langle f_\ell^m | L^2 f_\ell^m \rangle - \langle f_\ell^m | L_z^2 | f_\ell^m \rangle \pm \hbar \langle f_\ell^m | L_z f_\ell^m \rangle$$
$$= \hbar^2 \ell(\ell+1) - \hbar^2 m^2 \pm \hbar^2 m$$

Finally: $|A_\ell^m| = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}$

which is also $\quad = \hbar\sqrt{(\ell \mp m)(\ell \pm m+1)}$

- At the top of the ladder: $L_+ f_\ell^\ell = 0 \Rightarrow A_\ell^\ell = 0$

  Indeed: $A_\ell^\ell = \hbar\sqrt{\ell(\ell+1) - \ell(\ell+1)} = 0$

- At the bottom of the ladder: $L_- f_\ell^{-\ell} = 0 \Rightarrow A_\ell^{-\ell} = 0$

  Indeed $A_\ell^{-\ell} = \hbar\sqrt{+\ell(\ell+1) + \ell(-\ell-1)} = 0$

$\boxed{\text{Pb } 4.19}$ Cannonical commutation relations

a) $\vec{L} = (L_x, L_y, L_z) = (yp_z - zp_y, zp_x - xp_z, xp_y - yp_x)$

using $[x_i, p_j] = i\hbar\delta_{ij}$

$$\begin{aligned}
[L_z, x] &= [xp_y - yp_x, x] \\
&= [xp_y, x] - [yp_x, x] \\
&= xp_yx - x^2p_y - yp_xx + xyp_x \\
&= \underbrace{xxp_y - x^2p_y}_{0} - y(p_xx - xp_x) \\
&= 0 + y[x, p_x] = i\hbar y
\end{aligned}$$

$$\boxed{[L_z, x] = i\hbar y}$$

Similarly, by permutation: $[L_z, y] = x[p_y, y] = -i\hbar x$

and $[L_z, z] = 0$

$$
\begin{aligned}
[L_z, P_x] &= [xP_y - yP_x, P_x] \\
&= [xP_y, P_x] - [yP_x, P_x] \\
&= x\underbrace{P_y P_x} - P_x x P_y - \underbrace{yP_x P_x + P_x y P_x}_{0} \\
&= x P_x P_y - P_x x P_y \\
&= [x, P_x]\, P_y = +i\hbar P_y
\end{aligned}
$$

Similarly, by permutation

$$
\begin{aligned}
[L_z, P_y] &= [xP_y - yP_x, P_y] \\
&= -[y, P_y] P_x \\
&= -i\hbar P_x
\end{aligned}
$$

And $[L_z, P_z] = [xP_y - yP_x, P_z] = 0$

b) $[L_z, L_x] = [L_z, yP_z - zP_y] = [L_z, yP_z] - [L_z, zP_y]$

$$
\begin{aligned}
&= [L_z, y] P_z - z[L_z, P_y] \\
&= -i\hbar x P_z + i\hbar z P_x = i\hbar (zP_x - xP_z) = i\hbar L_y
\end{aligned}
$$

$\Rightarrow \boxed{[L_z, L_x] = i\hbar L_y}$

Similarly by permutation $[L_x, L_y] = i\hbar L_z$

$[L_y, L_z] = i\hbar L_x$

Pb 4.19 c) $[L_z, r^2] = [L_z, x^2+y^2+z^2]$

$$= [L_z, x^2] + [L_z, y^2] + \underbrace{[L_z, z^2]}_{0}$$

$$= L_z x^2 - x^2 L_z + L_z y^2 - y^2 L_z$$

$$= \cancel{xL_zx} + i\hbar yx - \cancel{xL_zx} + i\hbar xy + (-i\hbar xy - i\hbar yx)$$

$$= i\hbar(yx+xy) - i\hbar(xy+yx) = 0$$

$$\boxed{[L_z, r^2] = 0}$$

$$[L_z, p^2] = [L_z, p_x^2 + p_y^2 + p_z^2]$$

$$= [L_z, p_x^2] + [L_z, p_y^2] + \underbrace{[L_z, p_z^2]}_{0}$$

$$= 2i\hbar p_y p_x \cancel{+} 2i\hbar p_x p_y = 0$$

$$\boxed{[L_z, p^2] = 0}$$

d) Hamiltonian $H = \frac{p^2}{2m} + V$ (where V depends on $\|\vec{r}\|$ only)

operator r^2

$$[L_z, H] = \frac{1}{2m}\underbrace{[L_z, p^2]}_{0} + [L_z, V]$$

+ Combination $\underbrace{[L_z, r^2]}_{0}$

$[L_z, H] = 0$

Similarly $[L_x, H] = [L_y, H] = 0$

L_z, L^2, H are all compatible observables

(L_x and H) are compatibles and L_y and H are compatible.

$\boxed{\text{Pb 4.21}}$ a) $L_{\pm} = \pm \hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$

$$L_+L_- = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Using a test function f:

$$(L_+L_-)f = -\hbar^2 e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left[e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right)\right]$$

$$= -\hbar^2 e^{i\phi}\left[e^{-i\phi}\left(\frac{\partial^2 f}{\partial\theta^2} + i\csc^2\theta\frac{\partial f}{\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial\theta\partial\phi}\right) + i\cot\theta\left[(-i)e^{-i\phi}\left(\frac{\partial f}{\partial\theta} - i\cot\theta\frac{\partial f}{\partial\phi}\right) + e^{-i\phi}\left(\frac{\partial^2 f}{\partial\theta\partial\phi} - i\cot\theta\frac{\partial^2 f}{\partial^2\phi}\right)\right]\right]$$

$$= -\hbar^2\left[\frac{\partial^2 f}{\partial\theta^2} + i(\csc^2\theta - \cot^2\theta)\frac{\partial f}{\partial\phi} + \underbrace{(-i\cot\theta + i\cot\theta)}_{0}\frac{\partial^2 f}{\partial\theta\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} - i^2\cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right]$$

with $\csc^2\theta - \cot^2\theta = 1$

$$(L_+L_-)f = -\hbar^2\left[\frac{\partial^2 f}{\partial\theta^2} + i\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial f}{\partial\theta} + 1\times\cot^2\theta\frac{\partial^2 f}{\partial\phi^2}\right]$$

$$\Rightarrow \boxed{L_+L_- = -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right]}$$

b) L_+L_- are linked to L^2 and L_z by:

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$\begin{cases} L_+L_- = -\hbar^2\left(\dfrac{\partial^2}{\partial\theta^2} + \cot\theta\dfrac{\partial}{\partial\theta} + \cot^2\theta\dfrac{\partial^2}{\partial\phi^2} + i\dfrac{\partial}{\partial\phi}\right) \\ L_z = \dfrac{\hbar}{i}\dfrac{\partial}{\partial\phi} \quad , \quad L_z^2 = -\hbar^2\dfrac{\partial^2}{\partial\phi^2} \end{cases}$$

$$\Rightarrow L^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial^2}{\partial\phi^2} - \frac{\hbar^2}{i}\frac{\partial}{\partial\phi}$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + (\cot^2\theta + 1)\frac{\partial^2}{\partial\phi^2}\right)$$

$$= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

noticing that $\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) = \dfrac{1}{\sin\theta}\left(\cos\theta\dfrac{\partial}{\partial\theta} + \sin\theta\dfrac{\partial^2}{\partial\theta^2}\right)$

$$= \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}.$$

$$\Rightarrow \boxed{L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}$$

$\boxed{\text{Pb } 4.22}$

a) $L_+ Y_\ell^\ell = \phi$ because Y_ℓ^ℓ is at the top of the ladder

b) $L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)$

$$L_+ Y_\ell^\ell = 0 \Rightarrow \frac{\partial Y_\ell^\ell}{\partial\theta} + i\cot\theta \frac{\partial Y_\ell^\ell}{\partial\phi} = 0 \quad (1)$$

Separation of variables $Y_\ell^\ell = f(\theta)\, g(\phi)$

$$L_z Y_\ell^\ell = \hbar\ell\, Y_\ell^\ell \Rightarrow \frac{\hbar}{i}\frac{\partial g}{\partial\phi} = \hbar\ell\, g(\phi) \Rightarrow g(\phi) = e^{i\ell\phi}$$

$$\Rightarrow Y_\ell^\ell(\theta,\phi) = e^{i\ell\phi} f(\theta)$$

Back to (1) ; $\frac{df}{d\theta} + i\cot\theta\,(i\ell)\, f = 0$

$$\Rightarrow \frac{df}{d\theta} = \ell\cot\theta\, f \quad \Rightarrow \frac{df}{f}(\theta) = \ell\cot\theta\, d\theta$$

$$\Rightarrow \ln f = \ell\int\cot\theta\, d\theta = \ell\ln(\sin\theta)$$

$$\Rightarrow f = \sin^\ell\theta$$

$$\boxed{Y_\ell^\ell(\theta,\phi) = A\left(e^{i\phi}\sin\theta\right)^\ell}$$

-, From normalization: $\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} |Y_\ell^\ell|^2 \sin\theta\, d\theta\, d\phi = A^2\int_0^\pi 2\pi \sin^{2\ell+1}\theta\, d\theta = 1$

$$1 = 2\pi A^2 \times 2\frac{(2^\ell \ell!)^2}{(2\ell+1)!} \Rightarrow \boxed{A = \frac{1}{2^{\ell+1}\ell!}\sqrt{\frac{(2\ell+1)!}{\pi}}}$$