Physics 451- Fall 2012

Homework #17

Due Tuesday, Nov. 6 by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
We will have help sessions twice a week:
Tuesday session (with Jacob): from 2 pm to 5pm – room 363 MARB
Thursday session (with Peter): from 3pm to 6pm – room 393 CB

grading Total 40

List of problems (from the textbook):

4.6
Spherical well
4.7
4.8
4.9 (solve for bound state only)

4.6 — 10 pts
4.7 — a) 4 pts (2 for n_1, 2 for n_2) b) 6 pts
4.8 — a) 4 pts b) 6 pts
4.9 — 10 pts (6 for general solution + continuity, 4 for derivation on E, V_0)

Hints:

For problem 4.6, the trick is to integrate by-part l times in order to bring all the derivation on the same term. You should get l boundary terms plus on final integral. You can show that all the boundaries terms are zero because they all contain at least one factor (x^2-1) which becomes zero when you take the value at x =-1 and x =1. You then need to take care of the final integral. Discuss the three cases $l > l'$, $l < l'$ (same result by symmetry) and $l = l'$. Remember that deriving an n^{th} order polynomial more than n times gives zero. For $l = l'$, we will need to calculate the integral. Use a change of variable x = *cos* θ

useful formulae: $\dfrac{d^n}{dx^n}\left(x^n\right) = n!$ and $\displaystyle\int_0^\pi \sin^{2l+1}\theta d\theta = 2\frac{2*4*6....(2l)}{1*3*5...(2l+1)} = 2\frac{\left(2^l l!\right)^2}{(2l+1)!}$

For problem 4.9: solve for a bound state only, write the solution inside and outside the spherical well. Use the continuity of u and du/dr at the boundary $r = a$. Show that these conditions are equivalent to solving an equation like $\sqrt{\left(\dfrac{z_0}{z}\right)^2 - 1} = -\tan^{-1}(z)$.

Discuss in which case there is a solution.

$\boxed{\text{Homework \# 17}}$

$\boxed{\text{Pb 4.6}}$ $$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

$$\int_{-1}^{+1} P_\ell(x)\, P_{\ell'}(x)\, dx = \int_{-1}^{+1} \frac{1}{2^\ell \ell!}\,\frac{1}{2^{\ell'}\ell'!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \left(\frac{d}{dx}\right)^{\ell'} (x^2-1)^{\ell'} dx$$

$$= \frac{1}{2^{\ell+\ell'}}\,\frac{1}{\ell!}\,\frac{1}{\ell'!}\underbrace{\int_{-1}^{+1} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \,\frac{d^{\ell'}}{dx^{\ell'}}(x^2-1)^{\ell'} dx}$$

integration by parts : ℓ times until having all the derivation on same term

$$= \frac{1}{2^{\ell+\ell'}}\,\frac{1}{\ell!}\,\frac{1}{\ell'!}\left(\left[\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \,\frac{d^{\ell'}}{dx^{\ell'}}(x^2-1)^{\ell'}\right]_{-1}^{+1} - \int_{-1}^{+1} \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \,\frac{d^{\ell'+1}}{dx^{\ell'+1}}(x^2-1)^{\ell'} dx\right)$$

$$= \frac{1}{2^{\ell+\ell'}}\,\frac{1}{\ell'!}\,\frac{1}{\ell!}\left(\underbrace{\text{boundary terms}}_{0} \dots + (-1)^\ell \int_{-1}^{+1} (x^2-1)^\ell \,\frac{d^{\ell'+\ell}}{dx^{\ell'+\ell}}(x^2-1)^{\ell'} dx\right)$$

- All the boundary terms are 0 because contain at least one factor (x^2-1) and taking the value at $x=\pm 1$ gives $(x^2-1)=0 \Rightarrow$ Boundary term $=0$

- Now, for the last term, the $(x^2-1)^{\ell'}$ is a polymial in order $2\ell'$

If $\boxed{\ell > \ell'}$ then $\frac{d^{\ell+\ell'}}{dx^{\ell+\ell'}}(x^2-1)^{\ell'} = 0$ because $\ell+\ell' > 2\ell$

If $\boxed{\ell < \ell'}$ we just integrate by part the other way and find that same result.

Finally if $\boxed{l = l'}$, we have

$$\int_{-1}^{+1} P_l(x)\, P_l(x)\, dx = \frac{1}{2^{2l}(l!)^2}(-1)^l \int_{-1}^{+1} (x^2-1)^l \frac{d^{2l}}{dx^{2l}}(x^2-1)^l\, dx$$

But $\frac{d^{2l}}{dx^{2l}}(x^2-1)^l$ just leaves the first factor $= (2l)!$

$$\Rightarrow \int_{-1}^{+1} |P_l(x)|^2 dx = \frac{(-1)^l (2l)!}{2^{2l}\, l!\, l!} \int_{-1}^{+1} (x^2-1)^l\, dx$$

The integral can be done by change of variables:

let $X = \cos\theta \Rightarrow x^2 - 1 = -\sin^2\theta$ and $dx = -\sin\theta\, d\theta$

$$\Rightarrow \int_{-1}^{+1} (x^2-1)^l\, dx = \int_{\theta=\pi}^{\theta=0} (-\sin^2\theta)^l(-\sin\theta)\, d\theta = (-1)^l\, 2\int_0^{\pi/2} \sin^{2l+1}\theta\, d\theta$$

But $\displaystyle\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}\theta\, d\theta = \frac{(n-1)(n-3)\cdots}{n(n-2)\cdots}\,\frac{2}{1}\times$

$$\text{So } \int_0^{\pi} \sin^{(2l+1)}\theta\, d\theta = \frac{(2l)(2l-2)\cdots 2}{(2l+1)(2l-1)\cdots 1} = \frac{2^l l!}{\left((2l+1)!/2^l l!\right)} = \frac{(2^l l!)^2}{(2l+1)!}$$

$$\Rightarrow \int_{-1}^{+1} |P_l(x)|^2 dx = \frac{(-1)^l (2l)!}{2^{2l}\, l!\, l!}(-1)^l\, 2\frac{(2^l\, l!)^2}{(2l+1)!} = 2\,\frac{2^{2l}}{2^{2l}}\frac{(2l)!}{(2l+1)!} = \frac{2}{2l+1}$$

$$\Rightarrow \boxed{\int_{-1}^{+1} P_l(x)\, P_{l'}(x)\, dx = \frac{2}{2l+1}\,\delta_{ll'}}$$

$\boxed{\text{Pb } 4.7}$ $\mathcal{N}_\ell(x) = (-x)^{\ell+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell} \frac{\cos x}{x}$ Spherical Neumman function

a)

$$\mathcal{N}_0(x) = -\frac{\cos x}{x}$$

$$\mathcal{N}_1(x) = +x\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{\cancel{x}}{x}\left(\frac{-\sin x}{x} - \frac{\cos x}{x^2}\right) = -\frac{\sin x}{x} - \frac{\cos x}{x^2}$$

$$\mathcal{N}_2(x) = -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x}\right)\right) = -x^2\left(\frac{1}{x}\frac{d}{dx}\right)\left(-\frac{\sin x}{x^2} - \frac{\cos x}{x^3}\right)$$

$$= +x\frac{d}{dx}\left(\frac{\sin x}{x^2} + \frac{\cos x}{x^3}\right)$$

$$= +x\left(\frac{\cos x}{x^2} - \frac{2\sin x}{x^3} - \frac{\sin x}{x^3} - \frac{3\cos x}{x^4}\right)$$

$$\mathcal{N}_2(x) = -\frac{3\sin x}{x^2} + \frac{\cos x}{x}\left(1 - \frac{3}{x^2}\right)$$

b) When $x \ll 1$ $\sin x \simeq x - \frac{x^3}{3!}$ and $\cos x \simeq 1 - \frac{x^2}{2}$

(or to a first order: $\sin x \simeq x$ and $\cos x \simeq 1$)

$$\mathcal{N}_1(x) \simeq -\left(1 + \frac{1}{x^2}\right) \simeq -\frac{1}{x^2}$$ $\longrightarrow$ Blows up when $x \to 0$

$$\mathcal{N}_2(x) \simeq -\frac{3}{x} + \frac{1}{x}\left(1 - \frac{3}{x^2}\right) = -\frac{2}{x} - \frac{3}{x^3} \simeq -\frac{3}{x^3}$$ when $x \to 0$

$\mathcal{N}_2(x)$ Blows up (diverge) when $x \to 0$

$\boxed{\text{Pb } 4.8}$

a) Radial equation with $l = 1$ (for $u = rR$)

$$\frac{d^2u}{dr^2} = \left[\frac{1\times 2}{r^2} - k^2\right] u \qquad \text{with } k = \frac{\sqrt{2mE}}{\hbar}$$

Indeed: $u_1(r) = A_1 \, r \, j_1(kr)$

$$= A_1 \, r \left(\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right) \qquad \text{(see p. 142)}$$

$$= A_1 \left(\frac{\sin(kr)}{k^2 r} - \frac{\cos(kr)}{k}\right)$$

$$\frac{d^2u_1}{dr^2} = \frac{d}{dr}(A_1)\left(\frac{k\cos(kr)}{k^2 r} - \frac{\sin(kr)}{k^2r^2} + \frac{k\sin(kr)}{k}\right)$$

$$= A_1 \left(\frac{k}{k}\frac{(-\sin kr)}{r} - \frac{\cos kr}{kr^2} - \frac{k\cos(kr)}{k^2r^2} + \frac{2\sin(kr)}{k^2r^3} + k\cos(kr)\right)$$

$$= A_1 \left(\frac{\sin kr}{k^2 r}\left(\frac{2}{r^2} - k^2\right) - \frac{\cos(kr)}{k}\left(\frac{2}{r^2} - k^2\right)\right)$$

$$= \underbrace{A_1 \left(\frac{\sin kr}{k^2 r} - \frac{\cos(kr)}{k}\right)}_{u_1(r)}\left(\frac{2}{r^2} - k^2\right)$$

$$\Rightarrow \boxed{\frac{d^2u_1}{dr^2} = \left(\frac{2}{r^2} - k^2\right) u_1(r)}$$

4.8 (b) Allowed energies for the infinite spherical well, when $l = 1$

The radial solution is $R_{n1}(r) = \frac{u_1(r)}{r} = A_{n1} j_1(kr)$

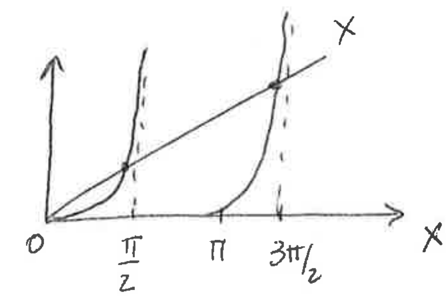
$$R_{n1}(r) = A_{n1}\left(\frac{\sin(kr)}{k^2r^2} - \frac{\cos(kr)}{kr}\right)$$

The condition at boundary impose that $R_{n1}(a) = 0$

$$\Rightarrow \quad \frac{\sin(ka)}{k^2a^2} - \frac{\cos ka}{ka} = 0$$

$$\Rightarrow \quad \frac{\sin(ka)}{ka} = \cos(ka) \quad \Rightarrow \quad \tan(ka) = ka$$

We need to solve for $\tan x = x$



For large X value: the solution is very close to $\frac{n\pi}{2}$

$$\Rightarrow \quad X \simeq \pi\left(n+\frac{1}{2}\right) \Rightarrow ka \simeq \left(n+\frac{1}{2}\right)\pi \Rightarrow k \simeq \left(n+\frac{1}{2}\right)\frac{\pi}{a}$$

$$\Rightarrow \quad \boxed{E_{n1} \simeq \frac{\pi^2\hbar^2}{2ma^2}\left(n+\frac{1}{2}\right)^2}$$

Pb 4.9 Finite spherical Well

$$V = \begin{cases} -V_0 & \text{if } r \leq a \\ 0 & \text{if } r > a \end{cases}$$

Radial equation: $\ell = 0$

$$\frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}(E+V_0)u$$ inside the sphere $r \lesssim a$

let $k' = \frac{\sqrt{2m(E+V_0)}}{\hbar}$ $\Rightarrow$ solution $u(r) = A\sin(k'r) + B\cos(k'r)$

at $r=0$ the $\frac{\cos(k'r)}{r}$ blows up so not physically possible.

Outside the sphere: $r > a$

$$\frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}Eu \quad , \quad k = \frac{\sqrt{2mE}}{\hbar}$$

if $E>0$ solution $u(r) = Be^{-ikr} + Ce^{ikr}$

$\Rightarrow$ $u(r) = A\sin(k'r)$

$R(r) = A\frac{\sin(k'r)}{r}$

Boundary conditions at $r=a$: the particle can still go through the boundary and exist outside the sphere

We still have the continuity of u and u' at the boundary:

$u_{r\leq a}(a) = u_{r>a}(a)$ and $u'_{r\leq a}(a) = u'_{r>a}(a)$

$$\begin{cases} A\sin(k'a) = B\sin(ka) + C\sin(ka) & (1) \\ k'A\cos(k'a) = k(-B\sin ka) + k(C\cos ka) & (2) \end{cases}$$

Combining (1) and (2)

$$A\left(k\sin(k'a)\sin(ka) + k'\cos(k'a)\sin(k'a)\right) = C\left(k\sin^2(ka) + k\cos^2(ka)\right)$$
$$= Ck$$

$\boxed{\text{Pb } 4.9}$ Finite spherical Well

$$V = \begin{cases} -V_0 & \text{if } r \leq a \\ 0 & \text{if } r > a \end{cases}$$

Radial equation : $\ell = 0$

Inside the sphere : $\frac{d^2 u}{dr^2} = -\frac{2m}{\hbar^2}(E + V_0) u$

Outside the sphere $\frac{d^2 u}{dr^2} = -\frac{2mE}{\hbar^2} u$

let's consider a bound state $-V_0 < E < 0$

- Solution inside the sphere $u(r) = A \sin(k' r) + B \cos(k' r)$ with $k' = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

physical condition $\Rightarrow$ $\frac{\cos(k' r)}{r}$ diverge at $r = 0$

$\Rightarrow u(r) = A \sin(k' r)$ for $r \leq a$

- Solution outside the sphere : $u(r) = e^{-kr}$ with $k = \sqrt{\frac{-2mE}{\hbar^2}}$

Condition at boundary : $r = a$

Continuity of u : $A \sin(k' a) = e^{-ka}$ (1)

Continuity of u' : $k' A \cos(k' a) = -k e^{-ka}$ (2)

Combining (1) and (2) $\Rightarrow$ $\boxed{\tan(k' a) = -\frac{k'}{k}}$

change of variable : $z = k' a$

$k'^2 = \frac{2m(E+V_0)}{\hbar^2}$, $k^2 = -\frac{2mE}{\hbar^2}$ $\Rightarrow$ $k'^2 = \frac{2mV_0}{\hbar^2} - k^2$ $\Rightarrow k^2 = \frac{2mV_0}{\hbar^2} - k'^2$

$\left(\frac{k}{k'}\right)^2 = \frac{2mV_0}{\hbar^2 k'^2} - 1$, $\frac{k}{k'} = \sqrt{\frac{2mV_0 a^2}{\hbar^2 z^2} - 1}$

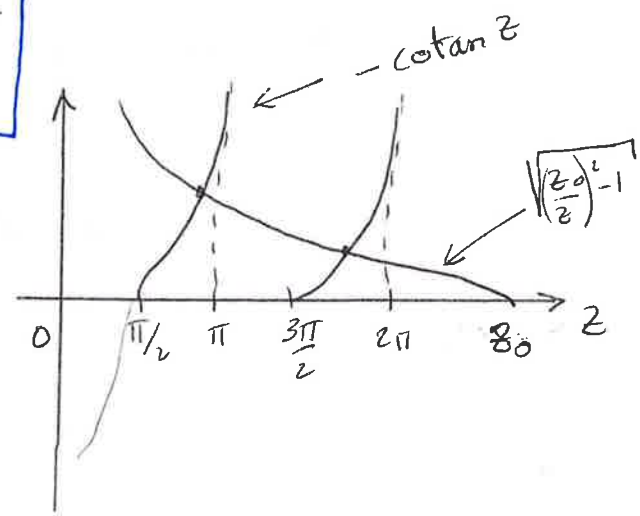
Calling $z_0 = \sqrt{\dfrac{2m V_0 a^2}{\hbar^2}}$

$$\Rightarrow \quad \frac{k}{k'} = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

$\Rightarrow$ We need to solve for $\tan z = -\dfrac{1}{\sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$

OR $$\boxed{\cot z = -\sqrt{\left(\frac{z_0}{z}\right)^2 - 1}}$$

Grapical solution (see pb 2.29)



We have a solution only if $z_0 \geq \frac{\pi}{2}$

$$\Rightarrow \quad \frac{2mV_0a^2}{\hbar^2} \geq \frac{\pi^2}{4}$$

$$\Rightarrow \quad \boxed{V_0 a^2 \geq \frac{\pi^2\hbar^2}{8m}}$$

In that case the ground state energy is given by the intersection between two graphs.

$$\frac{\pi}{2} \leq z \leq \pi$$

$$\frac{\pi}{2a} \leq k' \leq \frac{\pi}{a}$$

$$\frac{\pi^2}{4a^2} \leq \frac{2m(E+V_0)}{\hbar^2} \leq \frac{\pi^2}{a^2}$$

$$\boxed{-V_0 + \frac{\pi^2\hbar^2}{8ma^2} \leq E \leq -V_0 + \frac{\pi^2\hbar^2}{2ma^2}}$$

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<u>List of problems (from the textbook)</u>:

4.10
4.11
4.12
4.13

Hint:
For problem 4.10: the general method to find the functions R_{nl} is to find the coefficients of the power series c_0, c_1 c_2…expressing all them in terms of coefficient c_0

Useful integrals for Pb 4.11 and 4.13

$$\int_0^\infty x^n e^{-\alpha x}dx = \frac{n!}{\alpha^{n+1}}$$

For problem 4.13 (b): exploit symmetry of the ground state, knowing that the wave function Ψ_{100} has a spherical shape. In that state, the three axis (x, y, z) are equivalent, so averages values along these three directions are the same, and the absolute average position (<x>=<y>=<z>) is centered at the origin.

Grading

$\boxed{4.10}$ R_{30} 4 pts, R_{31} 3 pts, R_{32} 3 pts — 10

$\boxed{4.11}$ a) 5 pts b) 5 pts

$\boxed{4.12}$ a) 4 pts (L_0, L_1, L_2, L_3) b) 4 pts (derivation c) 2 pts

$\boxed{4.13}$ a) 4 (2 for $\langle r\rangle$, 2 for $\langle r^2\rangle$) b) 3 (1 for $\langle x\rangle = 0$, 2 for $\langle x^2\rangle$) c) 3

Total 40

$\boxed{\text{Homework \#18}}$

$\boxed{\text{Pb 4.10}}$

$\boxed{n=3}$ $k = \frac{1}{3a}$ 3 possibilities : $l=0$, $l=1$ and $l=2$

$\boxed{l=0}$ $c_0 \rightarrow c_1 = \frac{2(1-3)}{1 \times 2} c_0 = -2 c_0$

$$c_2 = \frac{2(2-3)}{2(3)} c_1 = -\frac{c_1}{3} = +\frac{2}{3} c_0$$

$$c_3 = 0$$

$$R_{30}(r) = \frac{1}{r} \rho e^{-\rho} \left(1 - 2\rho + \frac{2}{3}\rho^2\right) c_0 = c_0 \frac{kr}{r} e^{-kr}\left(1 - 2kr + \frac{2k^2}{3} r^2\right)$$

$$\boxed{R_{30}(r) = \frac{c_0}{3a}\left(1 - \frac{2r}{3a} + \frac{2r^2}{27a^2}\right) e^{-r/3a}}$$

$\boxed{l=1}$ $c_0 \rightarrow c_1 = \frac{2(2-3)}{1(4)} c_0 = -\frac{c_0}{2}$

$$c_2 = 0$$

$$R_{31}(r) = \frac{c_0}{r} \rho^2 e^{-\rho}\left(1 - \frac{\rho}{2}\right) = c_0 \frac{k^2 r^2}{r} e^{-kr}\left(1 - \frac{kr}{2}\right)$$

$$\boxed{R_{31}(r) = c_0 \frac{r}{9a^2}\left(1 - \frac{r}{6a}\right) e^{-r/3a}}$$

$\boxed{l=2}$ $c_0 \rightarrow c_1 = 0$

$$R_{32}(r) = \frac{c_0}{r} \rho^3 e^{-\rho} = \boxed{c_0 \frac{r^2}{27a^3} e^{-r/3a} = R_{32}(r)}$$

$\boxed{\text{Pb 4.11}}$

a) $R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right) e^{-r/2a}$

Namalization: $\int_{r=0}^{\infty} |R_{20}(r)|\, r^2 dr = 1$

$$\Rightarrow \frac{|c_0|^2}{4a^2}\int_0^{\infty} e^{-\frac{r}{a}}\left(1-\frac{r}{2a}\right)^2 dr = \frac{|c_0|^2}{4a^2}\int_0^{\infty} e^{-r/a}\left(1 - \frac{r}{a} + \frac{r^2}{4a^2}\right) r^2 dr$$

$$= \frac{|c_0|^2}{4a^2}\left(\int_0^{\infty} e^{-r/a} r^2 dr - \frac{1}{a}\int_0^{\infty} r^3 e^{-r/a} dr + \frac{1}{4a^2}\int_0^{\infty} r^4 e^{-r/a} dr\right)$$

$$= \frac{|c_0|^2}{4a^2}\left(2a^3 - \frac{a^4}{a}3! + \frac{4!}{4a^2}a^5\right) = \frac{|c_0|^2}{4a^2}a^3(2-6+6)$$

$$1 = |c_0|^2 a/2 \quad \Rightarrow \quad \boxed{c_0 = \sqrt{2/a}}$$

$$\boxed{\psi_{200}(r,\theta,\phi) = R_{20}(r)\, Y_0^0(\theta,\phi) = \frac{1}{\sqrt{2a^3}}\left(1 - \frac{r}{2a}\right) e^{-r/2a}\frac{1}{\sqrt{4\pi}}}$$

b) $R_{21}(r) = \frac{c_0}{4a^2} r e^{-r/2a}$

$$\int_0^{\infty} |R_{21}(r)|^2 r^2 dr = \frac{|c_0|^2}{16a^4}\int_0^{\infty} r^4 e^{-r/a} dr = \frac{c_0^2}{16a^4} a^5 4! = \frac{3}{2} a c_0^2$$

$$\Rightarrow \boxed{c_0 = \sqrt{\frac{2}{3a}}} \qquad \Rightarrow R_{21}(r) = \frac{1}{\sqrt{6a}}\frac{1}{2a^2} r e^{-r/2a}$$

We have 3 possible values for m: (see table for $Y_\ell^m(\theta,\phi)$)

$\underline{m=0}$: $\boxed{\psi_{210} = R_{21}(r)\, Y_1^0 = \frac{1}{\sqrt{2\pi a}}\frac{1}{4a^2} r e^{-r/2a} \cos\theta}$

$m = \pm 1$: $\boxed{\psi_{21\pm1} = R_{21}(r)\, Y_1^{\pm1}(\theta,\phi) = \mp \frac{1}{\sqrt{\pi a}}\frac{1}{8a^2} r e^{-r/2a} \sin\theta\, e^{\pm i\phi}}$

Pb4.12 Laguerre polynomials

a) $$L_q(x) = e^{x}\left(\frac{d}{dx}\right)^{q}\left(e^{-x}x^{q}\right)$$

$$L_0(x) = e^{x}\left(\frac{d}{dx}\right)^{0} e^{-x}x^{0} = e^{x}e^{-x} = 1$$

$$L_1(x) = e^{x}\frac{d}{dx}\left(e^{-x}x\right) = e^{x}\left(e^{-x} - xe^{-x}\right) = 1 - x$$

$$L_2(x) = e^{x}\frac{d^2}{dx^2}\left(e^{-x}x^2\right) = e^{x}\frac{d}{dx}\left(2x - x^2\right)e^{-x}$$

$$= e^{x}\left(2 - 2x - 2x + x^2\right)e^{-x}$$

$$L_2(x) = x^2 - 4x + 2$$

$$L_3(x) = e^{x}\frac{d^3}{dx^3}\left(e^{-x}x^3\right) = e^{x}\frac{d^2}{dx^2}\left(3x^2 - x^3\right)e^{-x}$$

$$= e^{x}\frac{d}{dx}\left(6x - 3x^2 - 3x^2 + x^3\right)e^{-x}$$

$$= e^{x}\left(+6 - 6x - 6x + 3x^2 - 6x + 6x^2 - x^3\right)e^{-x}$$

$$L_3(x) = -x^3 + 9x^2 - 18x + 6$$

b) $n = 5$, $l = 2$ $$v(\rho) = L^{4+1}_{5-2-1}(2\rho) = L^{5}_{2}(2\rho)$$

$$L^{5}_{2}(x) = L^{5}_{7-5}(x) = (-1)^{5}\left(\frac{d}{dx}\right)^{5}L_7(x)$$

$$L_7(x) = e^{x}\left(\frac{d}{dx}\right)^{7}\left(e^{-x}x^{7}\right) = e^{x}\frac{d^6}{dx^6}\left(7x^6 - x^7\right)e^{-x}$$

$$= e^{x}\frac{d^5}{dx^5}\left(7\times6\,x^5 - 7x^6 - 7x^6 + x^7\right)e^{-x}$$

$$= e^{x}\frac{d^4}{dx^4}\left((7\times5\times6)x^4 - (7\times6\times2)x^5 + 7x^6 - (7\times6)x^5 + (7\times2)x^6 - x^7\right)e^{-x}$$

$$= e^{x}\frac{d^3}{dx^3}\left(\left(7\times6\times5\times4\right)x^3 - \left(3\times7\times6\times5\right)x^4 + \left(3\times7\times6\right)x^5 - 7x^6 - (7\times6\times5)x^4 + 3(7\times6)x^5 - 3(7)x^6 + x^7\right)e^{-x}$$

$$L_7(x) = e^x \frac{d^3}{dx^3}\left(x^7 - (4\times 7)x^6 + 6(7\times 6)x^5 - (4)(7\times 6\times 5)x^4 + (7\times 6\times 5\times 4)x^3 \right) e^{-x}$$

$$= e^x \frac{d^2}{dx^2}\left(7x^6 - (7\times 6\times 4)x^5 + 6(7\times 6\times 5)x^4 - 4(7\times 6\times 5\times 4)x^3 + (7\times 6\times 5\times 4\times 3)x^2 \right.$$
$$\left. - x^7 + (4\times 7)x^6 - 6(7\times 6)x^5 + 4(7\times 6\times 5)x^4 - (7\times 6\times 5\times 4)x^3 \right) e^{-x}$$

$$= e^x \frac{d^2}{dx^2}\left(-x^7 + 5\times 7x^6 - 10(7\times 6)x^5 + 10(7\times 6\times 5)x^4 - 5(7\times 6\times 5\times 4)x^3 \right.$$
$$\left. + (7\times 6\times 5\times 4\times 3)x^2 \right) e^{-x}$$

$$= e^x \frac{d}{dx}\left(-7x^6 + (5\times 6\times 7)x^5 - 10(7\times 6\times 5)x^4 + 10(7\times 6\times 5\times 4)x^3 - 5(7\times 6\times 5\times 4\times 3)x^2 \right.$$
$$+ (7\times 6\times 5\times 4\times 3\times 2)x \qquad - (7\times 6\times 5\times 4\times 3)x^2$$
$$\left. + x^7 - 5\times 7x^6 + 10(7\times 6)x^5 - 10(7\times 6\times 5)x^4 + 5(7\times 6\times 5\times 4)x^3 \right) e^{-x}$$

$$= e^x \frac{d}{dx}\left(x^7 - (6\times 7)x^6 + 15(7\times 6)x^5 - 20(7\times 6\times 5)x^4 + 15(7\times 6\times 5\times 4)x^3 \right.$$
$$\left. - 6(7\times 6\times 5\times 4\times 3)x^2 + 7!\,x \right) e^{-x}$$

$$= e^x \left(7x^6 - 6(6\times 7)x^5 + 15(7\times 6\times 5)x^4 - 20(7\times 6\times 5\times 4)x^3 + 15(7\times 6\times 5\times 4\times 3)x^2 \right.$$
$$- 6\times 7!\,x + 7!$$
$$- x^7 + (6\times 7)x^6 - 15(7\times 6)x^5 + 20(7\times 6\times 5)x^4 - 15(7\times 6\times 5\times 4)x^3$$
$$\left. + 6(7\times 6\times 5\times 4\times 3)x^2 - 7!\,x \right) e^{-x}$$

$$L_7(x) = -x^7 + 7^2 x^6 - (21\times 7\times 6)x^5 + 35(7\times 6\times 5)x^4 - 35(7\times 6\times 5\times 4)x^3$$
$$+ 21(7\times 6\times 5\times 4\times 3)x^2 - 7\times 7!\,x + 7!$$

$$L_7(x) = \left(-x^7 + 7\times\frac{7!}{6!}x^6 - 21\frac{7!}{5!}x^5 + 35\frac{7!}{4!}x^4 - 35\frac{7!}{3!}x^3 + 21\frac{7!}{2!}x^2 - 7\,7!\,x + 7! \right)$$

Pb 4.13 (continued)

c) $n = 2 \;,\; l = 1 \;,\; m = 1$

$$\psi_{211}(r,\theta,\phi) = R_{21}(r)\, Y_1^1(\theta,\phi)$$

$$= \frac{1}{2\sqrt{6a^3}} \frac{r}{a} e^{-r/2a} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{i\phi}$$

$$= \frac{1}{2\sqrt{16\pi a^3}} \frac{r}{a} e^{-r/2a} \sin\theta\, e^{i\phi}$$

$$\psi_{211}(r,\theta,\phi) = \frac{1}{8} \frac{1}{\sqrt{\pi a^3}} \frac{r}{a} e^{-r/2a} \sin\theta\, e^{i\phi}$$

$$\langle x^2 \rangle = \langle \psi_{211} | x^2 | \psi_{211} \rangle \qquad x = r\sin\theta\cos\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |\psi_{211}|^2 x^2\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} |\psi_{211}|^2\, r^4 dr \sin^3\theta\, d\theta\, \cos^2\phi\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \cos^2\phi\, d\phi \int_{\theta=0}^{\pi} \sin^5\theta\, d\theta \int_{r=0}^{\infty} \frac{1}{8^2\pi a^3} \frac{r^2}{a^2} \times r^4 e^{-r/a} dr$$

$$= \frac{1}{64\pi a^5} \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^{\pi} \sin^5\theta\, d\theta \int_0^{\infty} r^6 e^{-r/a} dr$$

$$= \frac{1}{64\pi a^5} \left(\frac{2\pi}{2}\right) \left(2\,\frac{2\times 4}{1\times 3\times 5}\right) \left(6!\, a^7\right)$$

$$= a^2 \frac{6!}{8\times 8} \frac{4\times 4}{3\times 5} = a^2 \frac{6\times 5\times 4\times 3\times 2}{3\times 4\times 5} = 12a^2 \qquad \boxed{\langle x^2 \rangle = 12a^2}$$

$$L_7(x) = 7!\left(-\frac{x^7}{7!} + \frac{7x^6}{6!} - \frac{21x^5}{5!} + \frac{35x^4}{4!} - \frac{35x^3}{3!} + 21\frac{x^2}{2!} - 7x + 1\right)$$

$$L_2^5(x) = (-1)^5\left(\frac{d}{dx}\right)^5 L_7(x)$$

$$= (-1)^5\, 7!\left(-\frac{(7\times6\times5\times4\times3)}{7!}x^2 + \frac{7\times6\times5\times4\times3\times2}{6!}x - 21\,\frac{5\times4\times3\times2}{5!}\right)$$

$$= (-1)\, 7!\left(-\frac{7!}{7!\,2!}x^2 + \frac{7!}{6!}x - 21\times\frac{5!}{5!}\right)$$

$$= -\,7!\left(-\frac{x^2}{2} + 7x - 21\right)$$

$$\boxed{L_2^5(x) = 2(7!)\left(x^2 - 14x + 42\right)}$$

Finally: $V(\rho) = L_2^5(2\rho) = 2(7!)\left(4\rho^2 - 28\rho + 42\right)$

$$= 8\times7!\left(\rho^2 - 7\rho + \frac{21}{2}\right)$$

$$\boxed{V(\rho) = 8!\left(\rho^2 - 7\rho + \frac{21}{2}\right)}$$

c) From the recursion formula $n = 5$, $l = 2$

$$c_0 \to c_1 = \frac{2(3-5)}{6}c_0 = -\frac{2}{3}c_0$$

$$c_2 = \frac{2(4-5)}{2\times7}c_1 = -\frac{1}{7}c_1 = \frac{2}{21}c_0$$

$$c_3 = 2(5-5)c_2 = 0 \quad \text{stopped.}$$

$$\boxed{V(\rho) = c_0\left(1 - \frac{2}{3}\rho + \frac{2}{21}\rho^2\right) = \frac{2c_0}{21}\left(\rho^2 - 7\rho + \frac{21}{2}\right)}$$

same expression

(By Normalization of $R(r)$, we would find that $c_0 = 8!\times\frac{21}{2}$

$\boxed{\text{Pb 4.13}}$

Ground state of Hydrogen:

$$\psi_{100}(r,\theta,\varphi) = R_{10}(r)\, Y_0^0(\theta,\phi)$$

$$= \frac{2}{\sqrt{a^3}}\, e^{-r/a}\, \frac{1}{\sqrt{4\pi}} = \boxed{\frac{1}{\sqrt{\pi a^3}}\, e^{-r/a} = \psi_{100}(r)}$$

a) $$\langle r \rangle = \langle \psi_{100} | r | \psi_{100} \rangle$$

$$= \int_0^\infty |\psi_{100}|^2\, r \times r^2\, dr \underbrace{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta}_{4\pi} = \int_0^\infty e^{-2r/a}\, r^3\, dr \times \frac{4\pi}{\pi a^3}$$

$$= \frac{4\pi}{\pi a^3}\, 3! \left(\frac{a}{2}\right)^4 = 4\pi\, \frac{a \times 3!}{\pi \times 16} = \boxed{\frac{3a}{2} = \langle r \rangle}$$

$$\langle r^2 \rangle = \langle \psi_{100} | r^2 | \psi_{100} \rangle$$

$$= 4\pi \int_0^\infty |\psi_{100}|^2\, r^4\, dr = \frac{4\pi}{\pi a^3} \int e^{-2r/a}\, r^4\, dr$$

$$= \frac{4}{a^3}\, 4! \left(\frac{a}{2}\right)^5 = \frac{4!}{2^3}\, a^2 = \frac{4!}{8}\, a^2 = \frac{3 \times 2\, a^2}{2} = 3a^2$$

$$\boxed{\langle r^2 \rangle = 3a^2}$$

b) $\psi_{100}(r)$ is spherical, so x, y and z are equivalent

$\Rightarrow \langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle$ $\qquad \langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle = 3 \langle x^2 \rangle$

$$\Rightarrow \boxed{\langle x^2 \rangle = \frac{\langle r^2 \rangle}{3} = a^2}$$

Furthermore the density of probability is symmetrical about the origin

so $\boxed{\langle x \rangle = \langle y \rangle = \langle z \rangle = 0}$