Physics 451- Fall 2012

Homework #14
Due Thursday, Oct 18, by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
We have help sessions twice a week, in N337 ESC (undergraduate lab):
T Th from 4 to 6 pm

List of problems (from the textbook):

3.7
3.9
3.10
3.11
A26

Grading

3.7 a) 5, b) 5 — 10

3.9 a) 2, b) 2, c) 2 — 6

3.10 — 4 pts

3.11 — 10

A.26 a) 2, b) 2, c) 4, d) 2 — 10

Total 40

HOMEWORK #14

$\boxed{3.7}$

a) f and g two eigenstates of $\hat{Q}$ with same eigenvalue q

$$\left.\begin{array}{l}\hat{Q}|f\rangle = q\,|f\rangle \\ \hat{Q}|g\rangle = q\,|g\rangle\end{array}\right\} \quad \hat{Q}(\underbrace{\alpha|f\rangle + \beta|g\rangle}_{|\psi\rangle}) = \alpha q|f\rangle + \beta q|g\rangle = q\,\underbrace{(\alpha|f\rangle + \beta|g\rangle)}_{|\psi\rangle}$$

Any linear combination of $|f\rangle$ and $|g\rangle$ is eigenstaste of $\hat{Q}$ for q

b) $f(x) = e^{x}$, $g(x) = e^{-x}$

$\frac{d^2}{dx^2} f = f \longrightarrow f$ is an eigenvector of $\left(\hat{\frac{d^2}{dx^2}}\right)$ for eigenvalue $\boxed{1}$

$\frac{d^2}{dx^2} g = \frac{d}{dx}\left(-e^{-x}\right) = e^{-x} = g \longrightarrow g$ is also eigenvector for eigenvalue $\boxed{1}$

$$\begin{cases} \dfrac{f+g}{2} = \dfrac{e^{x}+e^{-x}}{2} = \cosh(x) & \text{is an eigenvector for value } \boxed{1} \\ \dfrac{f-g}{2} = \dfrac{e^{x}-e^{-x}}{2} = \sinh(x) & \text{is also an eigenvector for value } \boxed{1} \end{cases}$$

But $\int_{-1}^{+1} \cosh(x)\sinh(x)\,dx = 0$ (cosh is even, sinh is odd

$\Rightarrow$ $(\cosh)$ and $(\sinh)$ are two <u>orthogonal</u> eigenstates

2/6

3.9

a) Hamiltonian with discrete spectrum : - Infinite square well
(- harmonic oscillator)

b) Hamiltonian with only continuous spectrum (besides free particle)
is a potential for which we can ONLY have a scattering state $E>0$

- Delta-function barrier
- Finite rectangular barrier

c) Hamiltonian with both discrete and continuous spectra

discrete	continuous
$\downarrow$	$\downarrow$
$E<0$	$E>0$
Bound state	Scattering state

- Delta-function Well
- Finite Square Well

3.10

Ground state of infinite square well:

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right)$$

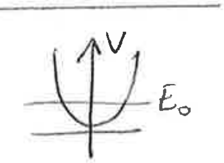
Operator momentum: $-i\hbar \frac{\hat{d}}{dx}$

$$-i\hbar \frac{d\psi_1}{dx} = -i\hbar \sqrt{\frac{2}{a}} \frac{d}{dx}\left(\sin\frac{\pi x}{a}\right) = -\frac{i\hbar\pi}{a}\sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)$$ NOT proportional to $\sin\left(\frac{\pi x}{a}\right)$

$\Rightarrow \psi_0$ is NOT an eigenfunction of operator momentum $\hat{p}$

3.11

Particle in the ground state of harmonic Oscillator



$$E_0 = \frac{\hbar\omega}{2} \qquad \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} e^{-i\omega t/2}$$

$\phi(p,t)$ is the Fourier Transform of $\psi_0(x,t)$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} e^{-\frac{m\omega x^2}{2\hbar}} dx$$

(projecting ψ_0 onto momentum eigenstate $e^{-\frac{ipx}{\hbar}}$)

$\boxed{3.11}$ (contd)

change of variable $u = x + ip/m\omega$

$u^2 = x^2 - \frac{2i}{m\omega} px - \frac{p^2}{m^2\omega^2}$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}\left(u^2 + \frac{p^2}{m^2\omega^2}\right)} du$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-i\omega t/2} e^{-\frac{p^2}{2m\hbar\omega}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{m\omega u^2}{2\hbar}} du}_{\sqrt{\frac{2\pi\hbar}{m\omega}}}$$

$$\phi(p,t) = \frac{e^{-\frac{p^2}{2m\hbar\omega}}}{(\pi m\hbar\omega)^{1/4}} e^{-i\omega t/2}$$

ϕ is also a Gaussian function

Probability of measuring p outside classical range is given by integrating $|\phi(p,t)|^2$ outside that range

Classical range: $E = \frac{p^2}{2m} + V \Rightarrow \frac{p^2}{2m} \leq E = \frac{\hbar\omega}{2}$

$\Rightarrow |p| \leq \sqrt{m\hbar\omega}$

$-\sqrt{m\hbar\omega} \leq p \leq \sqrt{m\hbar\omega}$

So $$\langle p \rangle = \int_{-\infty}^{-\sqrt{m\hbar\omega}} |\phi|^2 dp + \int_{+\sqrt{m\hbar\omega}}^{+\infty} |\phi|^2 dp$$

$$= 1 - \int_{-\sqrt{m\hbar\omega}}^{+\sqrt{m\hbar\omega}} |\phi|^2 dp$$

$$= 1 - \frac{1}{(\pi m\hbar\omega)^{1/2}} \int_{-\sqrt{m\hbar\omega}}^{\sqrt{m\hbar\omega}} e^{-\frac{p^2}{m\hbar\omega}} dp$$

But $\displaystyle\int_0^{\sqrt{m\hbar\omega}} |\phi|^2\, dp = \frac{1}{\sqrt{\pi m\hbar\omega}} \int_0^{\sqrt{m\hbar\omega}} e^{-p^2/m\hbar\omega}\, dp$

$$= \frac{1}{\sqrt{\pi m\hbar\omega}} \int_0^{\sqrt{m\hbar\omega}} e^{-\left(\frac{p}{\sqrt{m\hbar\omega}}\right)^2} dp$$

Change variable

$u = \dfrac{p}{\sqrt{m\hbar\omega}}$

$$= \frac{1}{\sqrt{\pi}} \int_0^1 e^{-u^2} du$$

$dp = \sqrt{m\hbar\omega}\; du$

So $\displaystyle \langle p \rangle = 1 - \frac{2}{\sqrt{\pi}} \underbrace{\int_0^1 e^{-u^2} du}_{0.3413} = 0.157 \simeq 0.16$

$\boxed{A26}$

$$T = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

a) $\det(T) = 2\,(4-1) + i\,(i+2i) + (-1-2)$

$= 2 \times 3 - 3 - 3 = 0$ $\qquad\boxed{\det T = 0}$

$\mathrm{Tr}(T) = 2+2+2 = 6$

b) $\det(T-\lambda I) = \begin{vmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2 - 1\right)$
$\qquad + i\left(i + (2-\lambda)i\right) + \left(-(2-\lambda) - 1\right)$

$= (2-\lambda)\left((2-\lambda)^2-1\right) - (3-\lambda) \quad \cancel{+3}+\lambda$

$= (2-\lambda)^3 - (2-\lambda) - (3-\cancel{\lambda}) + \lambda + \cancel{3}$

$= (2-\lambda)^3 + 3\lambda - 8$

$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 + 3\lambda - \cancel{8}$

$= -\lambda^3 + 6\lambda^2 - 9\lambda = -\lambda\left(\lambda^2 - 6\lambda + 9\right) = -\lambda(\lambda-3)^2$

$$\det\left(T-\lambda\mathbb{1}\right)=0 \quad\Rightarrow\quad \begin{cases}\lambda=0 & \text{eigenvalues}\\ \lambda=+3 & (\text{degenuacy } 2)\end{cases}$$

Sum $\lambda_1+\lambda_2+\lambda_3=6 = \mathrm{Tr}(T)$

in agreement with [A.85]

product $\lambda_1\lambda_2\lambda_3=0 = \det(T)$

Diagonalization: $T=\begin{pmatrix}0 & & 0\\ & 3 & \\ 0 & & 3\end{pmatrix}$

c) Eigevectors:

For $\lambda_1=0$: $T\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=0 \Rightarrow \begin{cases}2a+ib+c=0\\ -ia+2b+ic=0\\ a-ib+2c=0\end{cases}$

$\Rightarrow 3a+3c=0 \Rightarrow a=-c$

and $a+ib=0 \Rightarrow b=-ia \Rightarrow \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-i\\-1\end{pmatrix}=e_1$

For $\lambda_2=3$ $T\begin{pmatrix}a\\b\\c\end{pmatrix}=3\begin{pmatrix}a\\b\\c\end{pmatrix} \Rightarrow \begin{cases}2a+ib+c=3a\\ -ia+2b+ic=3b\\ a-ib+2c=3c\end{cases} \Rightarrow \begin{cases}-a+ib+c=0\\ -ia-b+ic=0\\ a-ib-c=0\end{cases}$

$\Rightarrow b=0$, $a=c$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}=e_2$

or $c=0$ and $a=+ib$ $(b=-ia)$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}=e_3$

(e_1, e_2, e_3) are not orthogonal, we need to orthogonalize with Gram-Schmidt orthogonalization procedure

Gram-Schmidt orthogonalization:

keep $e_1' = e_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ i\\ -1\end{pmatrix}$

then $e_2' = e_2 - \langle e_1 | e_2 \rangle e_1$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix} - 0$

keep $e_2' = e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$

then $e_3' = e_3 - \langle e_3 | e_2 \rangle e_2$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1/2\\ -i\\ -1/2\end{pmatrix}$

by normalization

$\Rightarrow e_3' = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\ -i\\ -1\end{pmatrix}$

This is $\underline{\underline{\text{one}}}$ possibility

d) "Diagonalization" matrix: $(e_1, e_2, e_3) \longrightarrow (e_1', e_2', e_3')$

$$S^{-1} = \begin{array}{cc} \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} & \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} & \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \end{array} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ i\sqrt{2} & 0 & -i \\ -\sqrt{2} & \sqrt{3} & -1 \end{pmatrix}$$

S is a unitary matrix: $S^{\dagger} = S^{-1}$ or $S = (S^{-1})^{\dagger}$

$$\Rightarrow S = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & -i\sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \\ 1 & i & -1 \end{pmatrix}$$

check that $STS^{-1} = \begin{pmatrix} 0 & & 0 \\ & 3 & \\ 0 & & 3 \end{pmatrix}$