Physics 451- Fall 2012

Homework # 13

Due Tuesday, Oct 16 by 7pm

Please place your assignment in the "Physics 451" slot across from N373 ESC.
We have help sessions twice a week, in N337 ESC (undergraduate lab):

T Th from 4 to 6 pm

List of problems (from the textbook):

3.3
3.5
A18
A19
A23
A25

grading total 50

3.3 — 5 pts	A 19 — 5 pts
3.5 — a) 6 pt (2 each) b) 2 pts c) 2 — 10	A 23 — a) 5 b) 5 — 10
A18 — 10 pts	A 25 — a) 2 pt b) 2 c) 2 d) 2 — 10

Hints

For Pb 3.3: develop the inner product in both forms for $h = f+g$ and for $h = f+ig$ and then combine the resulting equations.

Pb 3.5: Any operator has an hermitian conjugate, but the operator itself is called "Hermitian" only if $Q^\dagger = Q$. To find the hermitian conjugate use the integral form of

$$\langle g | Qf \rangle = \langle Q^\dagger g | f \rangle$$

For the raising operator, express a_+ in terms of operators p and x

For Pb A19: check how many eignevalues and eigenvectors could be found for M and conclude if the matrix M is diagonalizabe or not

For Pb A23: A matrix M is "normal" when: $\left[M, M^\dagger\right] = 0$

A matrix is diagonalizable when normal, but does not have to be normal to be diagonalizable.

Homework #13

Problem 3.3

Hermitian operator

$$\langle h | \hat{Q} h \rangle = \langle \hat{Q} h | h \rangle$$

If $h = f + g \Rightarrow \langle h | Q h \rangle = \langle f + g | Q (f+g) \rangle$

$$= \langle f | Q f \rangle + \langle f | Q g \rangle + \langle g | Q f \rangle + \langle g | Q g \rangle$$

So $\langle h | Q h \rangle = \langle Q f | f \rangle + \langle Q g | g \rangle + \langle g | Q f \rangle + \langle f | Q g \rangle$

but $\langle Q h | h \rangle = \langle Q f | f \rangle + \langle Q g | g \rangle + \langle Q f | g \rangle + \langle Q g | f \rangle$

Equaling the two sides: $\langle h | Q h \rangle = \langle Q h | h \rangle$

gives $\Rightarrow \langle g | Q f \rangle + \langle f | Q g \rangle = \langle Q f | g \rangle + \langle Q g | f \rangle \quad (1)$

Now using $h = f + ig$:

$$\langle h | Q h \rangle = \langle f | Q f \rangle + \langle g | Q g \rangle + i \langle f | Q g \rangle - i \langle g | Q f \rangle$$

$$= \langle Q f | f \rangle + \langle Q g | g \rangle + i \langle f | Q g \rangle - i \langle g | Q f \rangle$$

and $\langle Q h | h \rangle = \langle Q f | f \rangle + \langle Q g | g \rangle + i \langle Q f | g \rangle - i \langle Q g | f \rangle$

$$\Rightarrow \langle f | Q g \rangle - \langle g | Q f \rangle = \langle Q f | g \rangle - \langle Q g | f \rangle \quad (2)$$

Adding (1) and (2): $\Rightarrow 2 \langle f | Q g \rangle = 2 \langle Q f | g \rangle$

This what we want to show: $\boxed{\langle f | Q g \rangle = \langle Q f | g \rangle}$

2/9 $\boxed{\text{Pb } 3.5}$ Hermitian conjugate

a) $\boxed{\hat{x}^\dagger = \hat{x}}$ ($\hat{x}$ is hermitian)

Indeed $\langle g | \hat{x} f \rangle = \int_{-\infty}^{+\infty} g(x f)\,dx = \int_{-\infty}^{+\infty} (x g) f\,dx = \langle \hat{x} g | f \rangle$

$\boxed{\hat{i}^\dagger = -\hat{i}}$ ($\hat{i}$ is antihermitian)

Indeed $\langle g | \hat{i} f \rangle = \int_{-\infty}^{+\infty} g(i f)\,dx = \int_{-\infty}^{+\infty} (ig) f\,dx = -\langle ig | f \rangle$
$= \langle -ig | f \rangle$

$\boxed{\left(\widehat{\frac{d}{dx}}\right)^\dagger = -\left(\frac{d}{dx}\right)}$ (is anti hermitian)

For this we use integration by parts:

$\langle g | \frac{d}{dx} f \rangle = \int_{-\infty}^{+\infty} g \frac{df}{dx}\,dx = \underbrace{\left[g f \right]_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty} \frac{dg}{dx} f\,dx = -\langle \frac{dg}{dx} | f \rangle$

$= 0$ because must be squeare integrable

b) Harmonic oscillator Raising operator a_+

$\langle \psi | a_+ | \psi \rangle = \langle \psi | a_+ \psi \rangle = \langle (a_+)^\dagger \psi | \psi \rangle$

but $a_+ = \frac{1}{\sqrt{2\hbar m \omega}} (-i\hat{p} + m\omega \hat{x})$

$(a_+)^\dagger = \frac{1}{\sqrt{2\hbar m \omega}} \left(+i \hat{p}^\dagger + m\omega \hat{x}^\dagger \right) = \frac{1}{\sqrt{2\hbar m\omega}} \left(i\hat{p} + m\omega \hat{x} \right) = a_-$

So Finally! $\boxed{(a_+)^\dagger = a_-}$

$\boxed{\text{Pb } 3.5}$ (Continued)

c) $\langle f \mid (QR) g \rangle = \langle (QR)^\dagger f \mid g \rangle$

also $\langle f \mid (QR) g \rangle = \langle f \mid Q(Rg) \rangle$

$$= \langle Q^\dagger f \mid Rg \rangle$$

$$= \langle R^\dagger (Q^\dagger f) \mid g \rangle$$

$$= \langle R^\dagger Q^\dagger f \mid g \rangle$$

So $\boxed{(QR)^\dagger = R^\dagger Q^\dagger}$

$\boxed{\text{Pb A18}}$

Rotation $T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\det(T - \lambda I) = |T - \lambda I| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$\Rightarrow \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = 0$

$\boxed{1 + \lambda^2 - 2\lambda\cos\theta = 0}$ (characteristic equation)

$\Rightarrow$ Roots: $\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \dfrac{2\sqrt{-\sin^2\theta}}{2}$

$$= \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

These roots are complex except for $\theta = [0, \pi, 2\pi, \cdots]$

Then λ is real : $\lambda = \pm 1$

($\theta = \pi$) this corresponds to symmetry about the origin)

4/9

Finding the eigenvectors:

For $\lambda = e^{+i\theta}$ $\quad T|e_+\rangle = \lambda|e_+\rangle$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{i\theta}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha\cos\theta - \beta\sin\theta = e^{i\theta}\alpha = \cos\theta\,\alpha + i\sin\theta\,\alpha \\ \alpha\sin\theta + \beta\cos\theta = e^{i\theta}\beta = \cos\theta\,\beta + i\sin\theta\,\beta \end{cases}$$

$$\begin{cases} \alpha(i\sin\theta) + \beta\sin\theta = 0 \\ \Rightarrow \sin\theta(\beta + \alpha i) = 0 \\ \sin\theta(\alpha - \beta i) = 0 \end{cases} \Rightarrow \begin{cases} \beta + \alpha i = 0 \\ \alpha - \beta i = 0 \end{cases}$$

Solution: $\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha = 1 \\ \beta = -i \end{pmatrix}$ for $\lambda = e^{i\theta}$

Solution: $\frac{1}{\sqrt{2}}\begin{pmatrix} \alpha = 1 \\ \beta = i \end{pmatrix}$ for $\lambda = e^{-i\theta}$

Diagonalization of T:

$$\det S = \frac{1}{2}(2i) = i$$

$$S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & i \end{bmatrix} \quad \text{inverting} \Rightarrow S = \frac{1}{\det S}\tilde{C} \qquad \tilde{C} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$$

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$T^f = S\,T^{old}\,S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & +i \end{bmatrix}$$

$$T^f = \frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}\begin{bmatrix} \cos\theta - i\sin\theta & (\cos\theta - i\sin\theta) \\ \sin\theta - i\cos\theta & (\sin\theta + i\cos\theta) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (1+1)\cos\theta + (i+i)\sin\theta & (\cos\theta - \cos\theta) + i(\sin\theta - \sin\theta) \\ (\cos\theta - \cos\theta) - i(\sin\theta - \sin\theta) & (\cos\theta + \cos\theta) - i(\sin\theta + \sin\theta) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2(\cos\theta + i\sin\theta) & 0 \\ 0 & 2(\cos\theta - i\sin\theta) \end{bmatrix}$$

$$T^f = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

T^f is indeed diagonal in the form $T^f = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\boxed{\text{Pb A19}}$

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Diagonalization: possible if M is normal (~~or even~~ includes unitary or hermitian)

$$M^\dagger = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad M^\dagger M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$M^\dagger M - MM^\dagger \neq 0$, Also $\det(M - \lambda I) = 0$ gives only one eigenvalue $(\lambda = 1)$ and one eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This matrix is <u>not diagonalizable</u>.

Pb A 23

$$M = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}$$

a) A matrix is normal when $M^\dagger M - M M^\dagger = 0$

$$M^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} \quad \text{So } M^\dagger M = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 2 & 1+i \\ 1-i & 1+1 \end{pmatrix}$$

$$M M^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix}$$

$$M^\dagger M - M M^\dagger = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix} \neq 0$$

$[M^\dagger, M] \neq 0$ $\quad$ M is NOT normal!

b) $\det(M - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & i-\lambda \end{vmatrix} = (1-\lambda)(i-\lambda) - 1 = 0$

$$= \lambda^2 - (1+i)\lambda + (i-1) = 0$$

This specific equation has two roots:

$$\lambda_\pm = \frac{(1+i) \pm \sqrt{(1+i)^2 - 4(i-1)}}{2} = \frac{1+i \pm \sqrt{4-2i}}{2}$$

Since there are two distinct eigenvalues λ_+ and λ_- there are two independant eigenvectors!

$\Rightarrow$ M is digonalizable (in the forme of $\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$)

(even though M is NOT normal!)

Pb A25

$$T = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$$

(a) T is hermitian:

$$T^\dagger = \tilde{T}^* = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}^{\sim *} = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix} = T \qquad \boxed{T^\dagger = T}$$

b) Eigenvalues:

$$\det(T - \lambda I) = \begin{vmatrix} 1-\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - (1-i)(1+i)$$

$$= \lambda^2 - \lambda - (1 - i^2)$$

$$= \boxed{\lambda^2 - \lambda - 2 = 0}$$

Solutions: $\lambda = \dfrac{+1 \pm \sqrt{1+8}}{2} = \dfrac{1 \pm 3}{2}$ $\quad \boxed{\lambda_1 = 2, \ \lambda_2 = -1}$

c) Eigenvectors: $|e_1\rangle$ and $|e_2\rangle$

$$(T - \lambda_1)|e_1\rangle = 0 \Rightarrow \begin{pmatrix} 1-2 & 1-i \\ 1+i & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\alpha_1 + \beta_1(1-i) = 0 \\ \alpha_1(1+i) - 2\beta_1 = 0 \end{cases}$$

$\beta_1(1-i) = \alpha_1$ $\qquad$ Normalization $\alpha^2 + \beta^2 = 1$

$$\Rightarrow \beta^2 + |1-i|^2 \beta^2 = 1 \Rightarrow 3\beta^2 = 1$$

$$\Rightarrow \beta = \frac{1}{\sqrt{3}}$$

solution $|e_1\rangle = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$ for $\lambda_1 = 2$

$$(T - \lambda_2 I)|e_2\rangle = 0$$

$$\begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{cases} 2\alpha + (1-i)\beta = 0 \\ (1+i)\alpha + \beta = 0 \end{cases}$$

$$\Rightarrow \alpha = \frac{1-i}{2}\beta$$

Normalization $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow \left(\frac{|1-i|^2}{4} + 1\right)|\beta|^2 = 1$

$$\Rightarrow \beta = \sqrt{\frac{2}{3}} \Rightarrow \alpha = \frac{1-i}{\sqrt{6}}$$

$$\Rightarrow |e_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1-i \\ 2 \end{pmatrix} \quad \text{for } \lambda_2 = -1$$

d) Diagonalization matrix S

components of $|e_1\rangle$ and $|e_2\rangle$ in the old basis

$$S^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}}(1-i) & \frac{i-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \qquad S = \frac{1}{\det S}\tilde{C}$$

$$\det S = \frac{\sqrt{2}}{3}(1-i) - \frac{1-i}{3\sqrt{2}} = \frac{2(1-i) - (1-i)}{3\sqrt{2}} = \frac{(1-i)}{3\sqrt{2}}$$

But since S is unitary $S = (S^{-1})^{\dagger} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}$

So $$STS^{-1} = \frac{1}{3}\begin{pmatrix} 1+i & 1 \\ \frac{-(1+i)}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}\begin{pmatrix} 1-i & \frac{i-1}{\sqrt{2}} \\ 1 & \sqrt{2} \end{pmatrix}$$

$$S T S^{-1} = \frac{1}{3}\begin{pmatrix} 1+i & 1 \\ \frac{-(1+i)}{\sqrt{2}} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2-2i & \frac{1-i}{\sqrt{2}} + \sqrt{2}(i-1) \\ 2 & \sqrt{2} \end{pmatrix}$$

$$T^{f} = \frac{1}{3}\begin{pmatrix} 4+2 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$\uparrow$

This matrix is indeed digonal in the form of $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

e) $\det(T^f) = \lambda_1 \lambda_2 = -2$

$\mathrm{Tr}(T^f) = \lambda_1 + \lambda_2 = 1$

In the old form of T:

$\det(T) = -(1-i)(1+i) = -2$

$\mathrm{Tr}(T) = 1$

det (T) and Tr (T) are <u>independant of the basis.</u>