Phys 451 – sec 1, Fall 2010
Quantum Mechanics **CID______________**
Instructor: Karine Chesnel

Exam III

M Nov 22- Tu Nov 23, 2010

This test is time limited to 3 hours. The test is closed book and closed notes, but useful formulae and integrals are provided below. Please write your CID on each sheet of your work. When you are done, make sure to put your work in order and staple it. This test includes 5 problems equally weighted. Each problem will count toward your final score. I encourage you to first read through all questions in order to have a general idea of the test. Each of the five problems is focusing on specific topic of the course. You may answer the questions in the order you wish. Explain your reasoning as much as possible. Good luck!

Useful formulae and integrals

Schrödinger equation: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$

In spherical coordinates: $\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$

Potential in hydrogen atom: $V = -\frac{e^2}{4\pi\varepsilon_0 r}$ and Bohr radius: $a = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$

Wavevector in a bound state: $k = \sqrt{-2mE}/\hbar$

Rydberg formula: $\frac{1}{\lambda} = R\left|\frac{1}{n_f^2} - \frac{1}{n_i^2}\right|$ with $R = 1.097\times10^7 m^{-1}$

Normalization $\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta\int_0^{\infty} drr^2\psi_{nlm}^*\psi_{n'l'm'} = \delta_{nn'}\delta_{ll'}\delta_{mm'}$

$$\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta\left(Y_m^l\right)^* Y_{m'}^{l'} = \delta_{ll'}\delta_{mm'} \quad \text{and} \quad \int_0^{\infty} drr^2R(r) = 1$$

Potentially useful integrals: $\int_0^{\pi}\sin^2\theta d\theta = \frac{\pi}{2}$; $\int_0^{\pi}\sin^3\theta d\theta = \frac{4}{3}$; $\int_0^{\pi}\sin^{2l+1}\theta d\theta = 2\frac{\left(2^l l!\right)^2}{(2l+1)!}$;

$\int_0^{\pi}\sin\theta\cos\theta d\theta = 0$; $\int_0^{\pi}\sin\theta\cos^{2n+1}\theta d\theta = \int_0^{\pi}\sin^{2n+1}\theta\cos\theta d\theta = 0$; $\int_0^{\infty} r^n e^{-\alpha r}dr = \frac{n!}{\alpha^{n+1}}$

$|lm\rangle = Y_l^m(\theta,\phi)$; $L_z|lm\rangle = \hbar m|lm\rangle$; $L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle$

$L_\pm = L_x \pm iL_y$; $L_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l(m\pm1)\rangle$

Pauli matrices: $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $S_x = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

1. Hydrogen atom: spherical harmonics and angular momentum

The stationary states of the electron in the hydrogen atom are generally described by the wave function $\psi_{nlm}(r,\theta,\phi)$ in spherical coordinates.

a) What are the names of the three quantum number *n, l, m* and what possible values can they take? To which operators *n, l* and *m* are related respectively?

b) Using table 4.3, write down the spherical harmonic associated to the state $\psi_{431}(r,\theta,\phi)$

c) Show that the spherical harmonic $Y_3^1(\theta,\phi)$ is satisfying the angular part of the Schrödinger equation: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}=-l(l+1)Y$. When solving this equation, what are you actually looking for, in terms of the operator angular momentum?

d) Calculate $L^2Y_3^1$, $L_zY_3^1$, $L_+Y_3^1$, $L_-Y_3^1$, $L_xY_3^1$ and $L_yY_3^1$ in terms of spherical harmonics.

2. Hydrogen atom: radial function and energy

The electron of a hydrogen atom is in the stationary state ψ_{411}

a) Using the tables 4.3 and 4.7 write the wave function $\psi_{411}(r,\theta,\phi)$

b) Work out the radial part $R_{41}(r)$ using the recursion formula (up to a normalization constant):

$$R_{nl}(r)=\frac{1}{r}(kr)^{l+1}e^{-kr}v(kr) \qquad \text{where} \quad v(\rho)=\sum_{j=0}^{\infty}c_j\rho^j \quad \text{and} \quad c_{j+1}=\frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j$$

(Express your answer in terms of the wavevector *k*, *r* and c_0)

c) Comparing your results from a) and b), find the value of *k* in this state ψ_{411} and find the value of the energy *E* of the particle? (Express your answer in terms of the Bohr radius)

d) Assuming that *l* can only take integer values, what is the degeneracy of this energy level *E*?

e) If the electron would transit from the energy level E_4 to ground state E_2, what would be the wavelength of the light emitted? Where in the electromagnetic spectrum would it fall?

3. Electron's spin angular momentum and Pauli matrices

An electron is in the spin state $\chi_0 = A\begin{pmatrix}1\\2i\end{pmatrix}$ in the basis (χ_+,χ_-) (eigenspinors of S_z)

a) Find the normalization constant A

b) Using Pauli matrices, find the expectation values $\langle S_x\rangle$, $\langle S_y\rangle$ and $\langle S_z\rangle$

c) Calculate $\langle S_x{}^2\rangle$, $\langle S_y{}^2\rangle$, $\langle S_z{}^2\rangle$ and show that their sum is equal to what is expected for $\langle S^2\rangle$

d) When measuring S_z on the state χ_0, what is the probability of measuring $\hbar/2$?

e) When measuring S_x on the state χ_0, what is the probability of measuring $\hbar/2$?

4. Spin, magnetic field and precession

Let's consider an electron that is brought in the presence of a magnetic field applied in the z-direction: $\vec{B} = B_0\vec{k}$

a) With magnetic moment being $\vec{M} = \gamma\vec{S}$, shows that the Hamiltonian becomes $\hat{H} = -\gamma B_0 \hat{S}_z$. What is the name for γ? What are the possible values for the energy of the particle?

b) Write down the equation of motion (time-dependent Schrödinger equation) for the spin state $\chi(t)$.

c) If the initial state is $\chi_0 = a\chi_+^z + b\chi_-^z$ what is the state $\chi(t)$ at later times?

d) Calculate $\langle S_z \rangle$, $\langle S_x \rangle$ and $\langle S_y \rangle$. Do they depend on time?

e) According to your results in (d), what kind of motion the spin is following? If this motion is periodic, what is the frequency ω?

5. Combination of two spins

A particle with $S_1 = 2$ is combined with a particle with spin $S_2 = 1$

a) How many possible values can the spin S of the composite system take, and what are they?

b) Using the Clebsch-Gordan table 4.8 provided, write down the expression for the composite state $|sm\rangle = |31\rangle$ as a linear combination of the individual states $|s_1 m_1\rangle |s_2 m_2\rangle$

c) When measuring the spin $S_z^{(2)}$ of the second particle in the composite state $|sm\rangle = |31\rangle$, what is the probability of finding $+\hbar$ (meaning $m_2 = +1$)?

d) Apply the lowering operator $S_- = S_-^{(1)} + S_-^{(2)}$ to $|sm\rangle = |31\rangle$ and, using table 4.8, check that the result is indeed a scalar multiple of $|sm\rangle = |30\rangle$.