Exam II
Th Oct 27- Sat Oct 29, 2011

This test is time limited to 3 hours. The test is closed book and closed notes, but useful formulae are provided below. Please write your CID on each sheet of your work. When you are done, make sure to put your work in order and staple it. This test includes 5 problems equally weighted. Each problem will count toward your final score. I encourage you to first read through all questions in order to have a general idea of the test. Each of the five problems is focusing on specific topic of the course. You may answer the questions in the order you wish. Explain your reasoning as much as possible. Good luck!

Useful formulae

Schrödinger equation: \[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \]

Operators
\[ p = -i\hbar \frac{\partial}{\partial x} \quad H = \frac{p^2}{2m} + V \]

Delta function:
\[ \delta(x) = 0 \text{ for } x \neq 0; \quad \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \]

\[ \Delta \left( \frac{d\Psi}{dx} \right) = -\frac{2ma}{\hbar^2} \psi(0) \]

Trigonometry:
\[ \cos \alpha = \frac{1}{2} \left( e^{i\alpha} + e^{-i\alpha} \right) ; \quad \sin \alpha = \frac{1}{2i} \left( e^{i\alpha} - e^{-i\alpha} \right) ; \quad \cos^2 \alpha + \sin^2 \alpha = 1 \]
\[ \cosh \alpha = \frac{1}{2} \left( e^{\alpha} + e^{-\alpha} \right) ; \quad \sinh \alpha = \frac{1}{2} \left( e^{\alpha} - e^{-\alpha} \right) ; \quad \cosh^2 \alpha - \sinh^2 \alpha = 1 \]

The hermitian conjugate \( Q' \) of an operator \( Q \) is defined by: \[ \langle f | \hat{Q} g \rangle = \langle \hat{Q}' f | g \rangle \]

Generalized uncertainty principle
\[ \sigma_A^2 \sigma_B^2 \geq \left( \frac{[A,B]}{2i} \right)^2 \]
\[ \frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar} \left\langle [H,Q] \right\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle \]

1. Delta function well

Let’s consider a particle in a delta-function well potential \( V(x) = -\alpha \delta(x) \) where \( \alpha>0 \)

a) How many bound states and how many scattering states can be found in such potential? (Justify your answer)

b) The particle is in a bound state (\( E < 0 \)). Write the general solution for \( \psi \) in terms of \( k = \sqrt{-2mE/\hbar} \) in the relevant regions (you can use the notation \( \psi_c \) for \( x<0 \) and \( \psi_s \) for \( x>0 \))

c) Write the conditions at boundary (\( x=0 \)) for \( \psi \) and for \( d\psi/dx \). Use these conditions and the normalization of \( \psi \) to find the complete expression for \( \psi \)

d) Express the energy \( E \) in terms of \( \alpha \), \( m \) and \( \hbar \). (justify)
2. Finite Square barrier
Let’s consider a particle in a finite square barrier potential, for which:
\[ V = V_0 \text{ for } -a < x < a, \quad V = 0 \text{ elsewhere.} \]
The particle is in a scattering state with \( E < V_0 \)
a) Use the Schrödinger equation to find out the general shape of the wave function \( \psi(x) \) in the three regions \( x < -a, \quad -a < x < a \) and \( x > a \), in terms of \( k = \sqrt{2mE/\hbar} \) and \( l = \sqrt{2m(V_0-E)/\hbar} \)
b) Write the conditions at boundaries \((x=-a \text{ and } x=a)\) for \( \psi \) and for \( d\psi/dx \).
c) Combine the previous equations and show that the transmission coefficient \( T \) verifies:
\[ T^{-1} = 1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2 \left( \frac{2a}{\hbar} \sqrt{2m(V_0-E)} \right) \]
d) Draw the transmission coefficient \( T \) and the reflection coefficient \( R \) as function of energy \( E \) in the interval \([0, V_0]\).

3. Bras, kets and hermitian operators
a) Let’s consider two wave vectors \( f \) and \( g \) in the Hilbert space
Write the inner product \( \langle f \mid g \rangle \) in terms of an integral
b) Write the Schwarz inequality for \( f \) and \( g \) in terms of integrals
c) Assuming we can span the space with an orthonormal basis \( \left| e_n \right\rangle \), how would you write the projection operator onto \( \left| e_n \right\rangle \)? What is \( \sum_n \left| e_n \right\rangle \left\langle e_n \right| \) equal to?
d) An operator \( Q \) is hermitian if \( Q^\dagger = Q \). Show that the operators \( x \) and \( p \) are hermitian
e) Is the Hamiltonian \( H \) hermitian? Show why

4. Eigenvalues and eigenstates
Let’s consider the operator
\[ A = \begin{pmatrix} 2 & -i & 1 \\ i & 2 & -i \\ 1 & i & 2 \end{pmatrix} \]
a) Is the matrix \( A \) hermitian?
b) Determine the determinant of \( A \) and the trace of \( A \)
c) Find the eigenvalues of \( A \). Check that their sum and their product are consistent with (a)
d) Is \( A \) diagonalizable? If so, write down its diagonalized form.
e) Find the eigenvectors of \( A \). Within degenerate sector, construct linearly independent eigenvectors, so to get an orthogonal basis.
f) Construct the unitary matrix \( S \) that diagonalizes \( A \), and show explicitly how \( S \) reduces \( A \) to its diagonal form.

5. Generalized uncertainty principle
a) Using the generalized uncertainty principle provided, derive the Heisenberg’s uncertainty principle for \( x \) and \( p \)
b) Calculate the commutator \([x,H]\) and derive the uncertainty principle for \( x \) and \( H \)
c) Write the Heisenberg’s equation of motion (provided) for an operator \( Q \) that does not depend explicitly on time. Use this expression to derive the time – energy uncertainty principle. What are the definitions for \( \Delta t \) and \( \Delta E \) in this formalism?
d) Apply the Heisenberg equation of motion (provided) to the case of \( Q = p \). Show how your result is connected to the Ehrenfest’s theorem.