

Exam II
 Th Oct 28- Sat Oct 30, 2010

This test is time limited to 3 hours. The test is closed book and closed notes, but useful formulae are provided below. Please write your CID on each sheet of your work. When you are done, make sure to put your work in order and staple it. This test includes 5 problems equally weighted. Each problem will count toward your final score. I encourage you to first read through all questions in order to have a general idea of the test. Each of the five problems is focusing on specific topic of the course. You may answer the questions in the order you wish. Explain your reasoning as much as possible. Good luck!

Useful formulae

Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Delta function:

$$\delta(x) = 0 \text{ for } x \neq 0; \int_{-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Trigonometry: $\cos \alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})$; $\sin \alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$; $\cos^2 \alpha + \sin^2 \alpha = 1$

The hermitian conjugate Q^\dagger of an operator Q verifies: $\langle f | \hat{Q}g \rangle = \langle \hat{Q}^\dagger f | g \rangle$

Generalized uncertainty principle

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle [A, B] \rangle}{2i} \right)^2$$

Operators

$$p = -i\hbar \frac{\partial}{\partial x} \quad H = \frac{p^2}{2m} + V$$

Condition at boundaries for delta function

$$\Delta \left(\frac{d\psi}{dx} \right) = -\frac{2m\alpha}{\hbar^2} \psi(0)$$

Heisenberg equation of motion

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

Constants: $\hbar = 1.05 \times 10^{-34} \text{ Js}$; $m_e = 9.1 \times 10^{-31} \text{ kg}$; $1\text{eV} = 1.6 \times 10^{-19} \text{ J}$

1. Delta function potential

Let's consider a particle in a delta-function barrier $V(x) = \alpha\delta(x)$ where $\alpha > 0$

a) How many bound states and how many scattering states can be found in such potential?

(Justify your answer)

b) Let's consider a particle in a scattering state ($E > 0$). Solve the Schrödinger equation on both sides of the barrier, to find out the general shape of the particle's wave function $\psi(x)$ (stationary state) in terms of $k = \sqrt{2mE} / \hbar$

c) Use the conditions at boundary ($x = 0$) for ψ and for $d\psi/dx$, to determine the transmission coefficient $T = |\psi_{transmitted}|^2 / |\psi_{incident}|^2$ in terms of m , \hbar , α and E .

d) Plot T as a function of E . What is the probability of tunneling for an electron ($m = m_e$) with energy $E = 400 \text{ eV}$ if the barrier strength is $\alpha = 10^{-8} \text{ eV.m}$?

2. Finite square potential

Let's consider a particle in a finite square well: $V = -V_0$ for $-a < x < a$, $V = 0$ elsewhere.

The particle is in a bound state $-V_0 < E < 0$

- Use the Schrödinger equation to find out the general shape of the wave function $\psi(x)$ in the three regions $x < -a$, $-a < x < a$ and $x > a$, in terms of $k = \sqrt{-2mE} / \hbar$ and $l = \sqrt{2m(E + V_0)} / \hbar$
- Given the symmetry of the potential, what can we say about the symmetry of stationary states? Refine your solution $\psi(x)$ assuming it is an even solution.
- Write the conditions at boundaries ($x = -a$ and $x = a$) for ψ and for $d\psi/dx$.
- Use the conditions at boundaries to find an equation for the energy E , you may express your equation in terms of $z = la$ instead of E .
- What are the approximate values for the first energy levels E if the well is very deep ($V_0 \gg 1$) and wide ($a \gg 1$)? To what type of well known potential these energy levels correspond?

3. Bras, kets and hermitian operators

Let's consider two wave vectors f and g in the Hilbert space

- What kind of quantities are $\langle g | f \rangle$ and $|f\rangle\langle g|$?
- Write the Schwarz inequality for f and g in terms of integrals
- Assuming we can span the space with orthonormal basis $|e_n\rangle$, how would you write the projection operator onto $|e_n\rangle$? What is $\sum_n |e_n\rangle\langle e_n|$ equal to?
- An operator Q is hermitian if $Q^\dagger = Q$. Are the operators \hat{x} and \hat{p} hermitian? Justify

4. Eigenvalues and eigenstates

Let's consider the operators $A = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$

- Find the eigenvalues of A
- Find the eigenvectors of A
- Is A diagonalizable? If so write A in its diagonal form
- Is A hermitian? Explain.
- Calculate the determinant of A and the trace of A . Show the relationship between these quantities and the eigenvalues of A .

5. Generalized uncertainty principle

- Using the generalized uncertainty principle provided, derive the Heisenberg's uncertainty principle for the operators \hat{x} and \hat{p}
- Calculate the commutator $[\hat{x}, \hat{H}]$ and derive the uncertainty principle for \hat{x} and \hat{H}
- Write the Heisenberg's equation of motion (provided) for $\hat{Q} = \hat{x}$. What does it tell you? Use this expression to derive the time – energy uncertainty principle. What are the definitions for Δt and ΔE here?
- Apply the Heisenberg equation of motion (provided) to the case of $\hat{Q} = \hat{p}$. Show how your result is connected to the Ehrenfest's theorem.