Compton Scattering

1. Experiment

In the early 20th century, research into the interaction of X-rays with matter was an emerging field. It was observed that when a beam of X-rays is directed at an atom, an electron is ejected and the light is scattered through an angle \( \theta \), as illustrated below. Classical electromagnetism predicts that the wavelength of scattered rays should be equal to the initial wavelength; however, experiments showed that the wavelength of the scattered rays is actually greater than the initial wavelength. In 1923, Compton published a paper in the Physical Review explaining the phenomenon: A quantum theory of the scattering of X-rays by light elements (Phys. Rev. 21, 483 (1923)). Experimental measurements were also included in that paper, to support the theory.

![Compton Scattering diagram](image)

The Compton effect is essentially an *inelastic* scattering process between photons and electrons in the matter. This effect could not be explained by the classical theory of Thomson’s scattering – which basically describes an *elastic* scattering process. In the Compton effect, the inelastic scattering of photons in matter results in a decrease in energy (increase in wavelength) of an X-ray or gamma ray photon. Part of the energy of the X/gamma ray is transferred to a scattering electron, which recoils and is ejected from its atom, and the rest of the energy is taken by the scattered, "degraded" photon.

2. Classical theory of the phenomenon

In the classical view, Compton scattering can be simply described as a collision process between photons carrying the light and electrons in the matter. Invoking the particle – wave duality, the incident light is assimilated to a single photon of initial energy \( E = h\omega \) and initial momentum \( \vec{p} = \hbar \vec{k} \). After the “collision”, the photon is in a final state, with energy \( E' = h\omega' \) and final momentum \( \vec{p}' = \hbar \vec{k}' \). The initial and final wave vectors \( \vec{k} \) and \( \vec{k}' \) define the scattering plane, and the collision process can be reduced to a two-dimensional problem in that plane. The electron, initially at rest, acquires a final
momentum $\mathbf{p}_f$ and a final kinetic energy $E_f$. By applying the conservation laws from
Newtonian mechanics, one can obtain these relationships:

Conservation of momentum: $\mathbf{p}_f + \hbar \mathbf{k}' = \hbar \mathbf{k}$ \hspace{1cm} [1]
Conservation of energy: $E_f + \hbar \omega' = \hbar \omega$ \hspace{1cm} [2]

Using the De Broglie relationship $\omega = kc = 2 \pi c / \lambda$ \hspace{1cm} [3]
the conservation laws simplifies to the famous relationship:

$$\lambda' - \lambda = \frac{\hbar}{mc} (1 - \cos \theta)$$ \hspace{1cm} [4]

where $\theta$ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, also called the scattering angle.

It is interesting to note that since $E' \neq E$, the Compton effect is indeed an inelastic
scattering effect (contrarily to Thompson scattering which basically occurs at lower
energies and can be assimilated to an elastic scattering process).

3. Quantum theory

Coming back to the wave – particle duality, we will now describe both the photon and the
electron as wave functions. We need to translate the Compton interaction in terms of
Hamiltonian. Applying the Schrödinger equation (here Klein –Gordon equation) and
using the perturbation theory, we will then describe how the wave function of the
electron is affected by the interaction with the electromagnetic wave. We will then be
able to retrieve the conservation laws, and also to express the differential cross-section
for this Compton scattering process.

3.1 Klein – Gordon equation:
The Klein – Gordon equation, was introduced by Oskar Klein and Walter Gordon in 1927
to describe the behavior of relativistic electrons in the presence of electromagnetic
radiation, in the frame of quantum electrodynamics. It can be seen as the relativistic
equivalent of the Schrödinger equation:

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \hbar^2 c^2 \nabla^2 \Psi + m^2 c^4 \Psi$$ \hspace{1cm} [5]

Or, in the presence of an electromagnetic field

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = c^2 \left( -i \hbar \mathbf{\hat{V}} - q \mathbf{\bar{A}} \right)^2 \Psi + m^2 c^4 \Psi$$ \hspace{1cm} [6]

where $\mathbf{\bar{A}}$ represents the potential vector associated to the electromagnetic field.

In the case of Compton scattering, the potential vector results from the summation of the
incident field potential vector $\mathbf{\bar{A}}_0$ and the scattered field potential vector $\mathbf{\bar{A}}$:

$$\mathbf{\bar{A}} = \mathbf{\bar{A}}_0 + \mathbf{\bar{A}}_s$$
3.2 Interaction Hamiltonian

In the Klein-Gordon equation above, the Hamiltonian can be identified as:

\[ H = \frac{1}{m} (-i\hbar \nabla - q\vec{A})^2 + mc^2 \]  

[7]

When developing this Hamiltonian, and assuming that the scattering process is not going to affect substantially the wave function of the electron – or in other words, we can treat the scattering event as a perturbation - one can write: \( H = H_0 + H' \)

where \( H_0 = -\frac{\hbar^2}{m} \nabla^2 + 2i\frac{\hbar q}{m} \vec{A}_0 \cdot \nabla + \frac{q^2}{m} A_0^2 + mc^2 \) represents the unperturbed Hamiltonian

and \( H' = +2i\frac{\hbar q}{m} \vec{A}_0 \cdot \nabla + \frac{q^2}{m} A_0^2 + 2\frac{q^2}{m} \vec{A}_0 \cdot \vec{A}_0^\dagger \) represents the perturbation due to the scattering process.

Now we make the assumption that the first two terms (the gradient term and the quadratic term) are actually negligible in respect to the last term, so the perturbation can be written as:

\[ H' \approx 2\frac{q^2}{m} \vec{A}_0 \cdot \vec{A}_0^\dagger \]  

[8]

Finally, we need to find the expression for the potential vectors \( \vec{A}_0 \) and \( \vec{A}_s \). For the incident electromagnetic field: \( \vec{A}_0 = A_0 e^{i(k \cdot \vec{r} - \omega t)} \vec{\epsilon} \)  

[9]

For the scattered electromagnetic field: \( \vec{A}_s = A_s e^{i(k' \cdot \vec{r} - \omega' t)} \vec{\epsilon}' \)  

[10]

Where \( \vec{\epsilon} \) and \( \vec{\epsilon}' \) are the polarization of the respective radiations.

We can then write: \( H' \approx 2\frac{q^2 A_s A_0^*}{m} e^{i(k' - k) \cdot \vec{r} - (\omega' - \omega)t} \vec{\epsilon} \cdot \vec{\epsilon}^* \)  

[11]

3.3 Wave function for electron in the scattering process

We need to express the wave function of the electron and how it will be affected by the scattering process. For this, we assume that the electron is initially in a state that is not a bound state but technically a scattering state (in the Quantum Mechanics sense). The particle has an initial energy \( E_i \) and an initial momentum \( \vec{p}_i \) (this differs from the classical description, where the particle is initially at rest). As a result of the scattering process, the electron will be in a final state of energy \( E_f \) and momentum \( \vec{p}_f \). The main goal is to find a relationship between these quantities and deduct the differential cross-section of the Compton scattering process.

Since the electron is technically in a “scattering state” (in the QM sense), its wave function can be expressed as a linear superposition of a set of eigenstates spanning the Hilbert space. Similarly to the expansion presented in Griffiths’ textbook chapter 3,
equation (3.34) in one dimension, we can expand the wave function as following in three dimensions:

$$\Psi(\vec{r}, t) = \int c_p \Psi_p d^3 p$$ \hspace{1cm} [12]$$

where $\Psi_p$ are the eigenstates of the operator momentum and the coefficients $c_p$ correspond the probability of measuring the particle with a given momentum $\vec{p}$. Of course, in this description, we have a continuous spectrum of momentum or energy (no quantization), therefore the superposition turns into an integral.

In the context of relativistic quantum electrodynamics, and by applying the Klein-Gordon equation, one can show that the proper shape for the eigenfunctions is:

$$\Psi_p(\vec{r}, t) = \sqrt{\frac{mc^3}{(2\pi\hbar)^3}} e^{i(\vec{p}\cdot \vec{r} - E t)/\hbar}$$ \hspace{1cm} [13]$$

Next step is to find the coefficients $c_p$. The trick here is to expand these coefficients at different orders and use the perturbation theory. We have already written the Hamiltonian in terms of perturbation, in the paragraph 3.2. Similarly to the method presented in Griffiths’ textbook chapter 6.1 (except that we have here a continuum of states instead of a discrete set of states), one can find a relation ship between the zero order $c_p^{(0)}$ and the first order $c_p^{(1)}$:

$$c_p^{(1)} = -i\frac{mc^2}{2\hbar} \int dt \int d^3 \vec{p} \langle \Psi_p | H' | \Psi_p \rangle c_p^{(0)}$$ \hspace{1cm} [14]$$

Note, that since we have a time dependent Hamiltonian, there is here a time integration, in addition to the integration over momentum. (see Griffith Chapter 9).

Now, plugging in the expression for $H'$ and for $\Psi_p$ given above, we can show that

$$c_p^{(1)} = -i\frac{q^2 \pi mc^4 A_0 A_1 (\vec{e}, \vec{e}')}{2} \int d^3 \vec{p} \frac{\delta(E - E' + \hbar \omega - \hbar \omega') \delta(\vec{p} - \vec{p}' + \hbar \vec{k} - \hbar \vec{k}')}{\sqrt{E'E'}} c_p^{(0)}$$ \hspace{1cm} [15]$$

which finally reduces in

$$c_p^{(1)} = -2iq^2 \pi mc^5 A_0 A_1 (\vec{e}, \vec{e}') \frac{\delta(E - E' + \hbar \omega - \hbar \omega')}{\sqrt{E'E'}} c_p^{(0)}$$ \hspace{1cm} [16]$$

with the condition $\vec{p} - \vec{p}' + \hbar \vec{k} - \hbar \vec{k}' = 0$

Physically, the coefficient $c_p^{(1)}$ is associated to the particle in the final state, while the coefficient $c_p^{(0)}$ is associated to the particle in the initial state.
Here, we retrieve the conditions resulting from the conservation of energy and momentum, for the coefficient \( c_p^{(1)} \) to be non-zero:

\[
\begin{align*}
\mathbf{p}' + \hbar \mathbf{k}' &= \mathbf{p} + \hbar \mathbf{k} \\
E' + \hbar \omega' &= E + \hbar \omega
\end{align*}
\]  

But in addition, we also have some information about the “scattering rate” of this Compton scattering process, by exploiting the ratio:

\[
\left| \frac{c_p^{(1)}}{c_p^{(0)}} \right|^2 = \frac{q^2 \pi mc^4}{2} A_0 A_2 \left( \hat{\mathbf{e}} \cdot \hat{\mathbf{e}}' \right)^2
\]

\[\text{[19]}\]

### 3.4 Differential Cross-section

Using the formulae above, one can show that the resulting differential cross-section for the Compton scattering process is:

\[
\frac{d\sigma}{d\Omega_{\mathbf{k}'}'} = \left( \frac{q^2}{4\pi\varepsilon_0 mc^2} \right)^2 \left( \frac{k'}{k} \right)^2 |\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}'|^2
\]

\[\text{[20]}\]

So the total cross-section is:

\[
\sigma = \left( \frac{q^2}{4\pi\varepsilon_0 mc^2} \right)^2 \int \left( \frac{k'}{k} \right)^2 |\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}'|^2 d\Omega
\]

\[\text{[21]}\]

Noting that for transverse polarized light, the angle between \( \hat{\mathbf{e}} \) and \( \hat{\mathbf{e}}' \) is the same than between \( \hat{\mathbf{k}} \) and \( \hat{\mathbf{k}}' \), so \( \hat{\mathbf{e}} \cdot \hat{\mathbf{e}}' = \cos \theta \).