Physics 145: AC impedance and high-pass filters

What's the point? Become familiar with AC impedance concepts and apply them to the frequency response of a high-pass filter.

Equipment: LRC component board, LC meter, signal generator/oscilloscope stack, frequency meter, amplifier, cables.

INTRODUCTION

Resistance is commonly denoted by the letter *R*; and the SI unit for resistance is the **Ohm** (Ω). Under an the influence of an externally-applied potential difference *V*, a resistor will experience an electrical current that is related by the equation $R = V_R / I$**.** This principle is known as **Ohm's law**, and materials that obey it are referred to as **ohmic**. **Capacitance** is commonly denoted by the letter *C*; and the SI unit for capacitance is the **Farad** (F). Under an the influence of an externally-applied potential difference V_C, a capacitor will build up an electric charge related by $C = Q / V_C$**.** Because the effective current flowing through a capacitor is just the time derivative of the charge on the capacitor, we can also write $C = I /(dV_C / dt)$. The inductor is an important passive circuit component that you may not be familiar with. **Inductance** is commonly denoted by the letter *L*; and the SI unit for inductance is the **Henry** (H). Under an the influence of an externally-applied potential difference V_L, an inductor will experience a time-varying electric current related by $L = V_L /(dI / dt)$. Just as a stereotypical capacitor has the parallel plate geometry, the stereotypical inductor geometry is the multi-winding wire coil.

Impedance

For a **sinuosoidal-driving voltage** $V(t) = V_0\, e^{i\omega t} = |V_0| e^{i\phi} e^{i\omega t} = |V_0| e^{i(\omega t + \phi)}$, the electric current in any linear *R*, *L*, or *C* component responds with sinusoidal oscillations of the form $I(t) = I_0\, e^{i\omega t}$. Note that we have used a complex sinusoidal $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$, where ω is the angular frequency of the driving voltage, V_0 is its complex amplitude, and ϕ its phase. By convention, I_0 is often considered to be a real-valued quantity, so that the phase factor ϕ in the voltage amplitude reveals the extent to which the voltage and current oscillations are out of phase with one another. Under these conditions, the current-voltage relationships of resistors, capacitors, and inductors can all be generalized in terms of a common quantity known as **impedance**. The impedance of a cirucuit component, commonly denoted by the letter *Z*, is defined as the complex voltage amplitude across the component divided by the complex current amplitude through the component: $Z = V_0 / I_0$. Measured in units of Ohms, *Z* represents the extent to which the component resists or impedes the flow of current being driven by the external voltage. Employing on the derivative properties of the exponential function, one can take the current-voltage relationships that define resistance, capacitance and inductance, and quickly derive their respective impedances:

$$Z_R = R \qquad Z_L = i\omega L \qquad Z_C = \frac{1}{i\omega C}.$$

Even if you do not feel comfortable with the physical definitions of R, C and L, as described above, you will need to memorize these three impedance expressions and manipulate them extensively in the context of complex sinusoidal functions. Because the voltage across a resistor is always in phase with the current through the resistor, the real-valued quantity Z_R is nothing more than resistance. Notice, however, that Z_L and Z_C are purely imaginary-valued quantities, and that they depend on frequency. Z_L and Z_C are known as **reactance** rather than resistance, and produce a ±90° phase difference between current and voltage. For an inductive reactance, voltage leads current by 90°, which explains the factor of $i = e^{i\pi/2} = e^{i(90^\circ)}$ in Z_L. For a capacitive reactance, the voltage lags behind the current by 90°, which explains the factor of $-i = e^{-i\pi/2} = e^{i(-90^\circ)}$ in Z_C.

In the introductory electronics lab course, you learned how to add resistors and capacitors in series and in parallel. You may call that the rules for resistors and capacitors were somewhat different: series resistors add directly while series capacitors add inversely, and parallel resistors add inversely while parallel capacitors add directly. The rules for adding inductors are essentially the same as for resistors. These rules are summarized as follows:

$$\begin{array}{ll} R_{\text{series}} = R_1 + R_2 & 1/R_{\text{parallel}} = 1/R_1 + 1/R_2 \\ L_{\text{series}} = L_1 + L_2 & 1/L_{\text{parallel}} = 1/L_1 + 1/L_2 \\ 1/C_{\text{series}} = 1/C_1 + 1/C_2 & C_{\text{parallel}} = C_1 + C_2 \end{array}.$$

Upon generalizing to impedances, these three sets of rules are all described by a common set:

$$Z_{\text{series}} = Z_1 + Z_2 \qquad 1/Z_{\text{parallel}} = 1/Z_1 + 1/Z_2.$$

Any lumped behavior of any complicated network of resistors, inductors, and capacitors can be understood in terms of a total impedance that been built up from individual components using series and parallel addition.

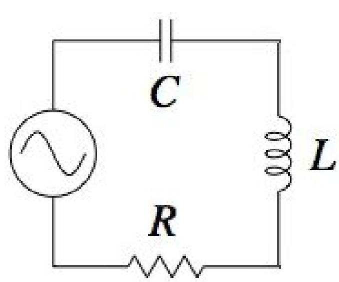


Figure 1. Driven series LRC circuit. The symbol at the left of the circuit is the AC voltage source that drives the circuit.

The total series impedance of a resistor, capacitor and inductor is the complex-valued quantity $Z = Z_R + Z_L + Z_C = R + i(\omega L - 1/\omega C)$, with magnitude $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ and phase $\tan(\phi) = (\omega L - 1/\omega C)/R$. For a given driving voltage, the magnitude of the current oscillations will be

$$I_0 = \left|\frac{V_0}{Z}\right| = \frac{|V_0|}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} .$$

AC Gain

If you think of the AC source as a voltage input with amplitude V_0, and the potential difference across some portion of your circuit as a voltage output, it is intuitive to define the voltage **gain** as $G = V_{out}/V_{in}$. This is just a generalization of the voltage divider concept. After choosing the portion of the circuit that forms the output, and determining its impedance, Z_{out}, one can compute the expected gain of a circuit as

$$G = \frac{V_{out}}{V_{in}} = \frac{I\,Z_{out}}{I\,Z} = \frac{Z_{out}}{Z} ,$$

where Z is the total impedance of the circuit as seen by the source. In the case of the series LRC circuit above, the output across the resistor (i.e. $Z_{out} = R$), provides a complex gain of

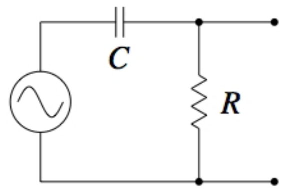
$$G = \frac{R}{R + i(\omega L - 1/\omega C)}, \text{where } |G| = \frac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \text{ and } \tan(\phi_G) = \frac{-(\omega L - 1/\omega C)}{R} .$$

PROCEDURE

Submit the complex-variables homework assignment. Feel free to ask your TA for help with a few of the problems. But don't spend more than 15 minutes working on it in class. You were expected to complete most of the assignment on your own before class began.

Measure and analyze the frequency response of a high-pass RC filter circuit.

1) For the high-pass RC filter in the diagram below, manually compute the following quantities as functions of cyclic frequency f and determine the cutoff frequency f_0: (**a**) Re part of Z, (**b**) Im part of Z, (**c**) magnitude of Z, (**d**) phase (argument) of Z, (**e**) magnitude of the gain $|Z_{out}/Z|$. For (**e**), try redrawing the circuit just as you would for a voltage divider, using impedances instead of resistances, and show that the voltage-divider equation yields the gain.



A filter has a frequency-dependent gain that is designed to pass certain frequencies and block others. It has a high gain at frequencies that should be passed (i.e.output comparable to input) and a low gain at frequencies that should be blocked (i.e. output much smaller than input). High and low-pass filters have a cutoff frequency that marks the transition between the pass and block regions, where the gain is equal to $\sqrt{1/2} = 0.707$. You can set your gain expression equal to this value and solve for f to obtain the cutoff frequency.

2) Use the Mathematica routine provided (*impedance.nb*) to repeat these calculations. Compare them to your manually-computed results to make sure that both are correct.

3) Using the components provided in the lab, design a high-pass RC filter which has a transition frequency somewhere between f_0 = 1.0 and 6.0 kHz. Use an R/L/C multimeter to check the actual values of each of your components, which may not agree with the values on the printed labels. After choosing circuit components, enter their values in *impedance.nb*, and plot each of the curves computed above over a frequency range from 10 Hz to 100 kHz. If you define the magnitude of the gain properly, it will approach a value of zero at one frequency extreme and a value of 1 at the other. Paste your plots into your lab notebook and discuss them with your TA and/or lab partner to ensure that they are correct and make *intuitive* sense.

4) Send a sine wave signal from the "High" output of your signal generator to your power amplifier (via the line input) and from there to the channel 1 input of your oscilloscope. Set your scope to trigger externally and obtain the trigger signal from the TTL output of your signal generator. Make sure that you have the DC bias of your signal generator "**OFF**" in order to avoid overheating your amplifier! This amplifier really hates a DC bias. Adjust your signal generator gain to the middle of its range, and then adjust the amplifier gain until the peak-to-peak signal from the power amplifier is 5 V. Use the frequency counter to obtain an accurate measure of your signal frequency. Be warned that the ground leads from each device should all be connected to each other, or else you will end up grounding your output, which results in a pretty small signal (i.e. zero). Have your TA review your progress at this point.

5) Now build the filter that you designed and use banana cables to drive it with the 5V signal from your amplifier (we call this the input signal). Let the resistor in your circuit be the output element and send its voltage to channel 2 of your oscilloscope. Simultaneously view the input and output signals on your scope. Sweep the entire frequency range from 0 to 100 kHz to get a qualitative feel for the behavior of the gain (i.e. the output to input ratio). Also

check to see if there is any frequency dependence in the amplitude of the input signal. If there is, you will need to account for this dependence when you measure the gain below.

6) Map out the frequency-dependent gain of your circuit from 100 Hz to 100 kHz in 30 uniform logarithmic increments of $\log_{10}(f)$. If the thought makes you go cross-eyed, define $a = \log_{10}(10^2) = 2$ and $b = \log_{10}(10^5) = 5$, use Excel to make a list of x values stretching from a to b in 30 uniform linear increments (31 data points), and then calculate $f = 10^x$ for each value in the list. Measure and record the magnitude $|G|$ of the gain at each frequency. Also estimate the approximate error in your gain measurements. Attach a frequency meter across the input in order to accurately measure frequency.

7) Create a three-column dataset in Logger Pro. Name the columns "Frequency (Hz)", "Gain" and "sigmaG". Graph the data as open circles with error bars and use a horizontal log scale. Fit a model of the form $|G| = \frac{A}{\sqrt{1+1/(2\pi f RC)^2}}$ to your data, using A and C as variables, but fixing R to the value measured using a multimeter. Note that A is an overall scale factor that compensates for the fact that your gain curve may not peak at exactly 1. If your model was entered correctly, your fit should converge easily. If the error bars are properly calculated, the fitted curve will typically stay within the error bars of most of the individual data points. Place the final graph, including error bars, the fitted curve, the fitting parameters and their uncertainties in your lab notebook.

8) Comment on how your fitted capacitance compares to the measured capacitance? Was the fitted value within one estimated standard deviation (i.e. the uncertainty from the fit) of the measured value? Using the fitted capacitance, compute the value of the 3db cutoff frequency, $f_{3db} = \frac{1}{2\pi RC}$. Demonstrate analytically that when $f = f_{3db}$, the decibel gain, $20\log_{10}(G)$, is approximately equal to -3.0. Hence the name f_{3db}.

9) **Extra Credit (1 pt)**: Briefly look through the whole range again to observe the behavior of the phase difference (phase of the input driving voltage minus the phase of the output current) over this same frequency range. Time runs from left to right across the oscilloscope screen. If the input peaks before the output in time, the phase is positive. If the output peaks first, the phase is negative. Note that the phase must stay within the range between -90° and +90°. A 90° phase corresponds to an input that peaks right where the output is rising through zero. Make a qualitative graph of the frequency-dependent phase for your lab notebook.