FRESNEL-KIRCHHOFF DIFFRACTION THEORY

- **Partial differential equation** approach:
  - ▷ Analytical solution
    - ∘ Solve PDE by separation of variables
    - ∘ Use boundary conditions to restrict set of possible solutions
    - ∘ Useful only for certain highly symmetrical geometries
  - ▷ Numerical solution:
    - ∘ Discretize using finite differences or finite elements
    - ∘ Obtain a system of linear equations that also incorporates the boundary conditions
- **Integral equation** approach:
  - ▷ Turn the PDE into an integral equation
  - ▷ Incorporate approximate boundary conditions
  - ▷ Get an approximate analytical solution
  - ▷ Numerical solution

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (1)

- Maxwell's equations imply the vector wave equation

$$\nabla \times [\nabla \times \mathbf{E}(\mathbf{r}, t)] + \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r}, t) = 0$$

  ▷ Vector (polarization) effects are often negligible except for very small apertures or obstacles

- In a uniform dielectric and in Cartesian coordinates,

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r}, t) = 0$$

- Scalar wave equation:

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)U(\mathbf{r}, t) = 0$$

  ▷ Valid for (at best) one Cartesian component of **E**

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (2)

- The scalar wave equation,

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) U(\mathbf{r}, t) = 0,$$

and the assumption of a perfectly monochromatic wave,

$$U(\mathbf{r}, t) = u(\mathbf{r})e^{-i\omega t},$$

imply the scalar Helmholtz equation

$$\boxed{\left(\nabla^2 + k^2\right) u(\mathbf{r}) = 0}$$

where

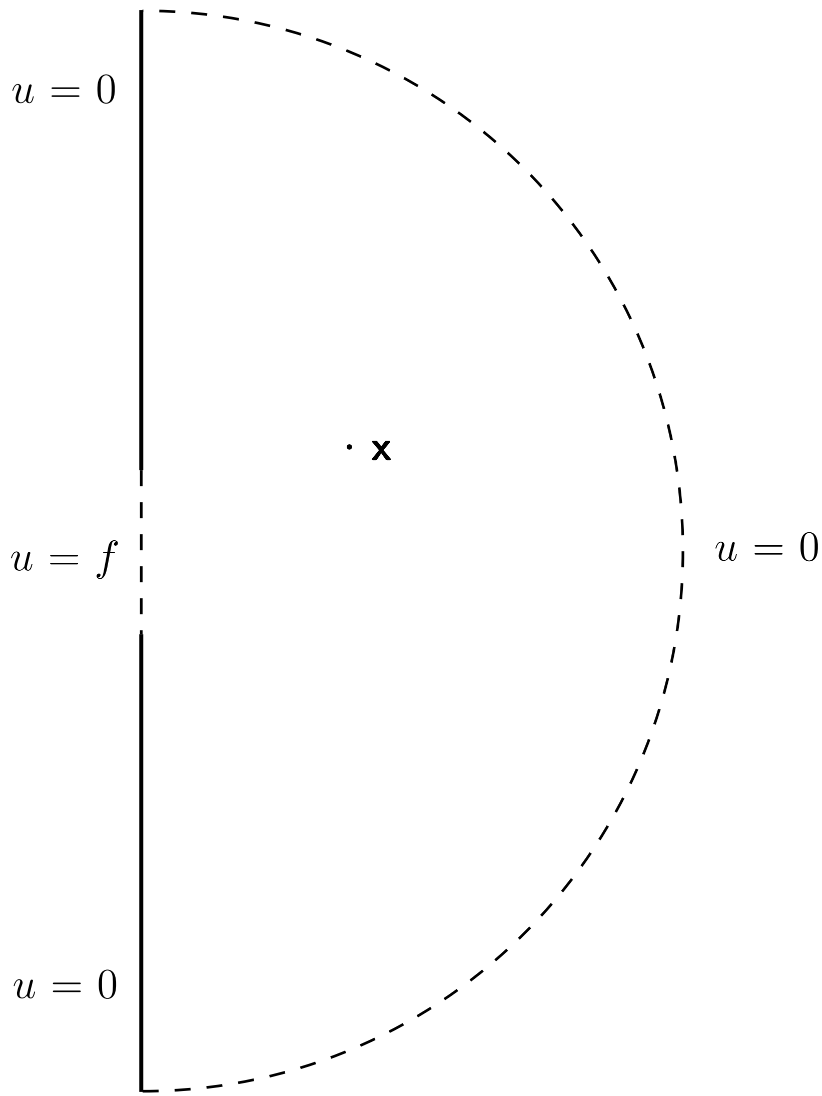
$$k^2 = \frac{\omega^2}{v^2}$$

▷ Boundary conditions for the Helmholtz equation

∘ Dirichlet (u given on the boundary)

∘ Neumann (du/dn given on the boundary)

DIRICHLET BOUNDARY CONDITIONS



BESSEL BEAMS

- Transverse radial profile is a Bessel function of the first kind
- **A Bessel beam propagates without spreading**
  - ▷ An exact solution of the Helmholtz equation $(\nabla^2 + k^2)u = 0$ is

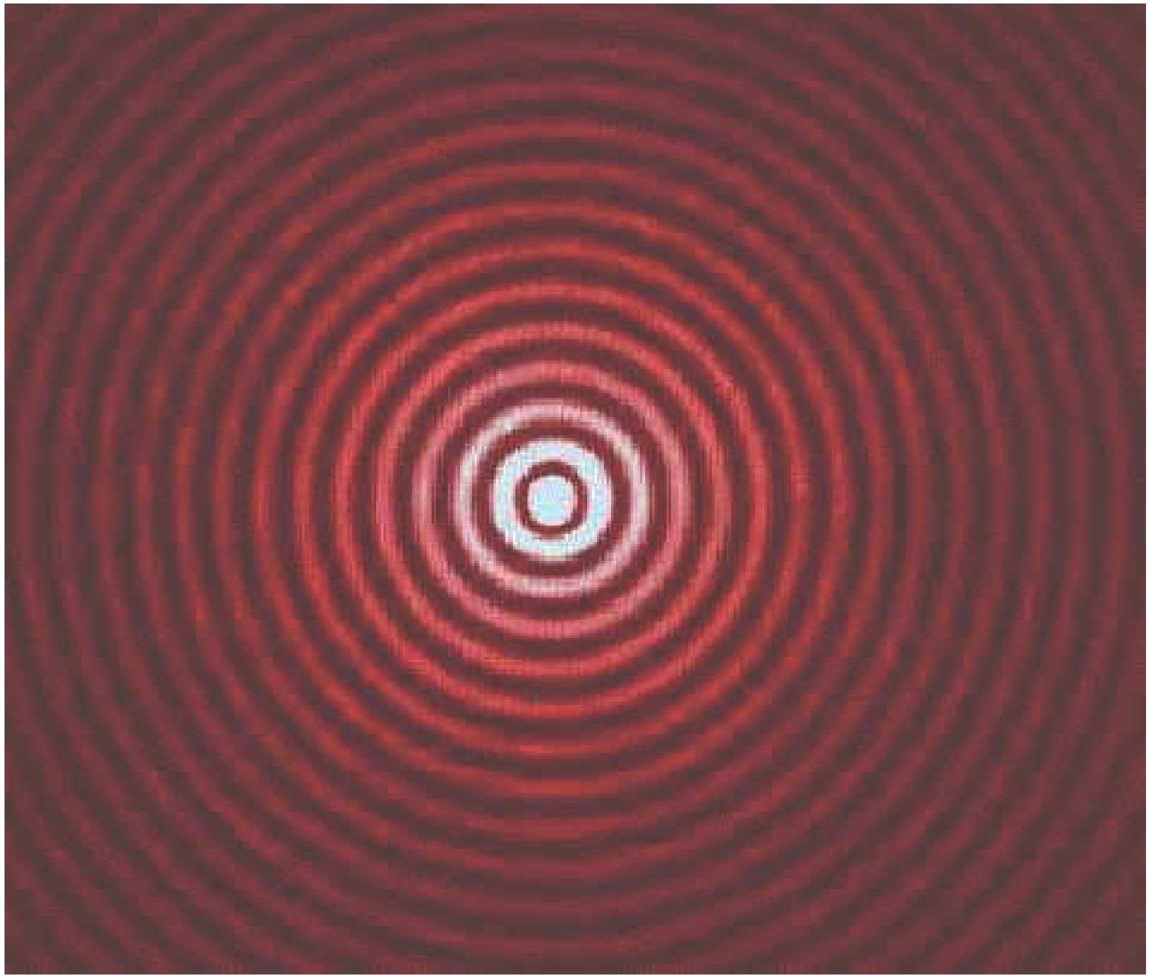
$$\boxed{u(\rho, \phi, z) = J_m(\alpha\rho)\, e^{im\phi}\, e^{i\beta z}}$$

  - ∘ The condition

$$k^2 = \alpha^2 + \beta^2$$

    must be satisfied
  - ∘ Most useful for $m = 0$, because J_0 gives the beam a central bright spot
  - ∘ Because the radial profile is independent of z, diffraction does not lead to spreading

INTENSITY PATTERN OF A BESSEL BEAM



http://www.doe.carleton.ca/∼rpm/bbessel.html

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (3)

- Approximate scalar boundary conditions obtained from **St.-Venant's hypothesis**:
  - ▷ The optical field u in an aperture is the same as if the aperture were not present
  - ▷ $u = 0$:
    - ◦ At all points on the screen
    - ◦ At large distances in image space
- Obviously not entirely correct, but useful
  - ▷ Ignores currents in the edges of the aperture

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (4)

- Green function for the scalar Helmholtz equation:
  - ▷ Scalar Helmholtz equation:

$$(\nabla^2 + k^2)u = 0$$

  - ▷ Green function in free space:

$$G(\mathbf{r}, \mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|}$$

    - ∘ G is a spherical wave expanding out from a point source at $\mathbf{r}$
    - ∘ G satisfies the equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r}-\mathbf{x})$$

      and outgoing boundary conditions at ∞
  - ▷ A solution of the equation $(\nabla^2 + k^2)u(\mathbf{r}) = -s(\mathbf{r})$ in a region V is

$$u(\mathbf{x}) = \begin{cases} (4\pi)^{-1} \int_V G(\mathbf{x}-\mathbf{r})\, s(\mathbf{r})\, d^3r, & \text{if } \mathbf{x} \in V; \\ 0, & \text{if } \mathbf{x} \notin V \end{cases}$$

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (5)

- Green function for the scalar Helmholtz equation:
  - ▷ Scalar Helmholtz equation:

$$(\nabla^2 + k^2)u = 0$$

  - ▷ Green function in free space (outgoing boundary conditions):

$$G(\mathbf{r}, \mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r} - \mathbf{x}|}$$

    - ○ G is a spherical wave expanding out from a point source at Q
    - ○ G satisfies the equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r} - \mathbf{x})$$

      and outgoing boundary conditions at ∞

  - ▷ A solution of the equation $(\nabla^2 + k^2)u(\mathbf{r}) = -s(\mathbf{r})$ is

$$u(\mathbf{x}) = \frac{1}{4\pi} \int G(\mathbf{x} - \mathbf{r})\, s(\mathbf{r})\, d^3r$$

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (6)

- Derivation of Green's theorem:
  - ▷ Recall Gauss's theorem:

$$\int_V \nabla \cdot \mathbf{F}\, d^3r = \int_{\partial V} (-\hat{\mathbf{n}}) \cdot \mathbf{F}\, dS$$

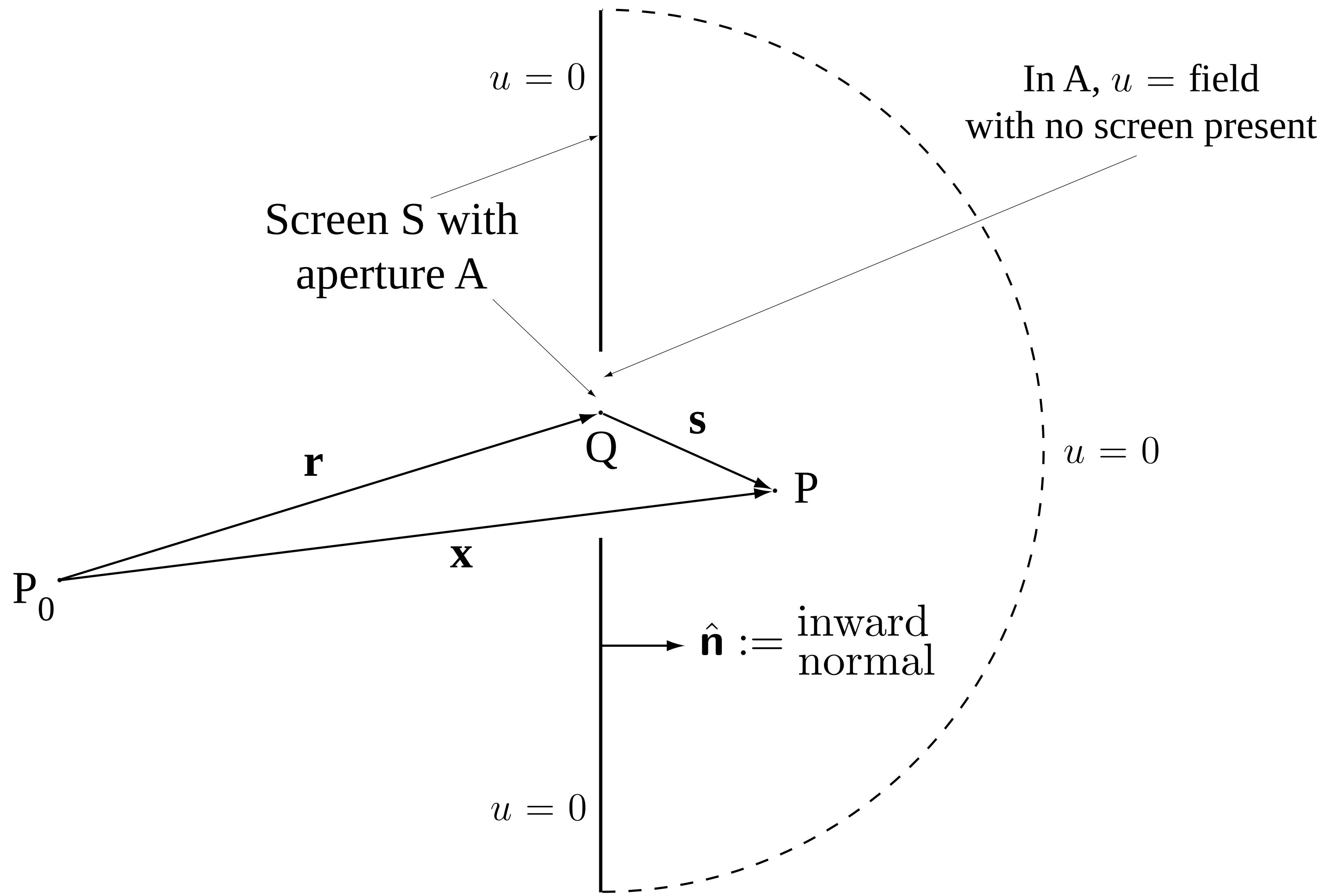
∂V := surface that bounds the volume V, $\hat{\mathbf{n}}$:= INWARD normal to ∂V

  - ▷ Apply Gauss's theorem to the vector field $\mathbf{F} = u\nabla G - G\nabla u$:

$$\begin{aligned}\int_{\partial V} (-\hat{\mathbf{n}})\cdot(u\nabla G - G\nabla u)\, dS &= \int_V \nabla \cdot (u\nabla G - G\nabla u)\, d^3r \\ &= \int_V [(\nabla u)\cdot(\nabla G) - (\nabla G)\cdot(\nabla u)]\, d^3r \\ &\quad + \int_V (u\nabla^2 G - G\nabla^2 u)\, d^3r \\ &= \int_V [u(\nabla^2 + k^2)G - G(\nabla^2 + k^2)u]\, d^3r = -4\pi u(\mathbf{x})\end{aligned}$$

    - ○ We used $(\nabla^2 + k^2)u = 0$ and $(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{x}) = -4\pi\delta(\mathbf{r} - \mathbf{x})$

KIRCHHOFF'S BOUNDARY CONDITIONS



FRESNEL-KIRCHHOFF DIFFRACTION THEORY (7)

- The **Helmholtz-Kirchhoff integral theorem**:
  - ▷ From Gauss's and Green's theorems,

$$u(\mathbf{x}) = \frac{1}{4\pi}\int_{\partial V} \left[u\,(\hat{\mathbf{n}}\cdot\nabla G) - G\,(\hat{\mathbf{n}}\cdot\nabla u)\right] dS$$

$\hat{\mathbf{n}}$:= INWARD normal to the surface that bounds V

$$G(\mathbf{r},\mathbf{x}) = \frac{e^{ik|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}-\mathbf{x}|}$$

  - ▷ Substitute for G:

$$u(\mathbf{x}) = \frac{1}{4\pi}\int_{\partial V} \left[u(\mathbf{r})\left(\hat{\mathbf{n}}\cdot\nabla\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\left(\hat{\mathbf{n}}\cdot\nabla u(\mathbf{r})\right)\right] dS$$

    - ∘ Expresses $u(\mathbf{x})$ in terms of fields at points $\mathbf{r}$ on the surface that bounds V, but not in the same way as Huyghens' Principle
    - ∘ Kirchhoff's boundary conditions apply

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (8)

- Kirchhoff's boundary conditions:
  - ▷ $u = 0$ everywhere but in the aperture
  - ▷ In the aperture, u is the field due to a point source at P_0,

$$u(\mathbf{r}) = A\,\frac{e^{ikr}}{r}$$

- Apply the Helmholtz-Kirchhoff integral theorem:
  - ▷ At any point $\mathbf{r}$ in the aperture,

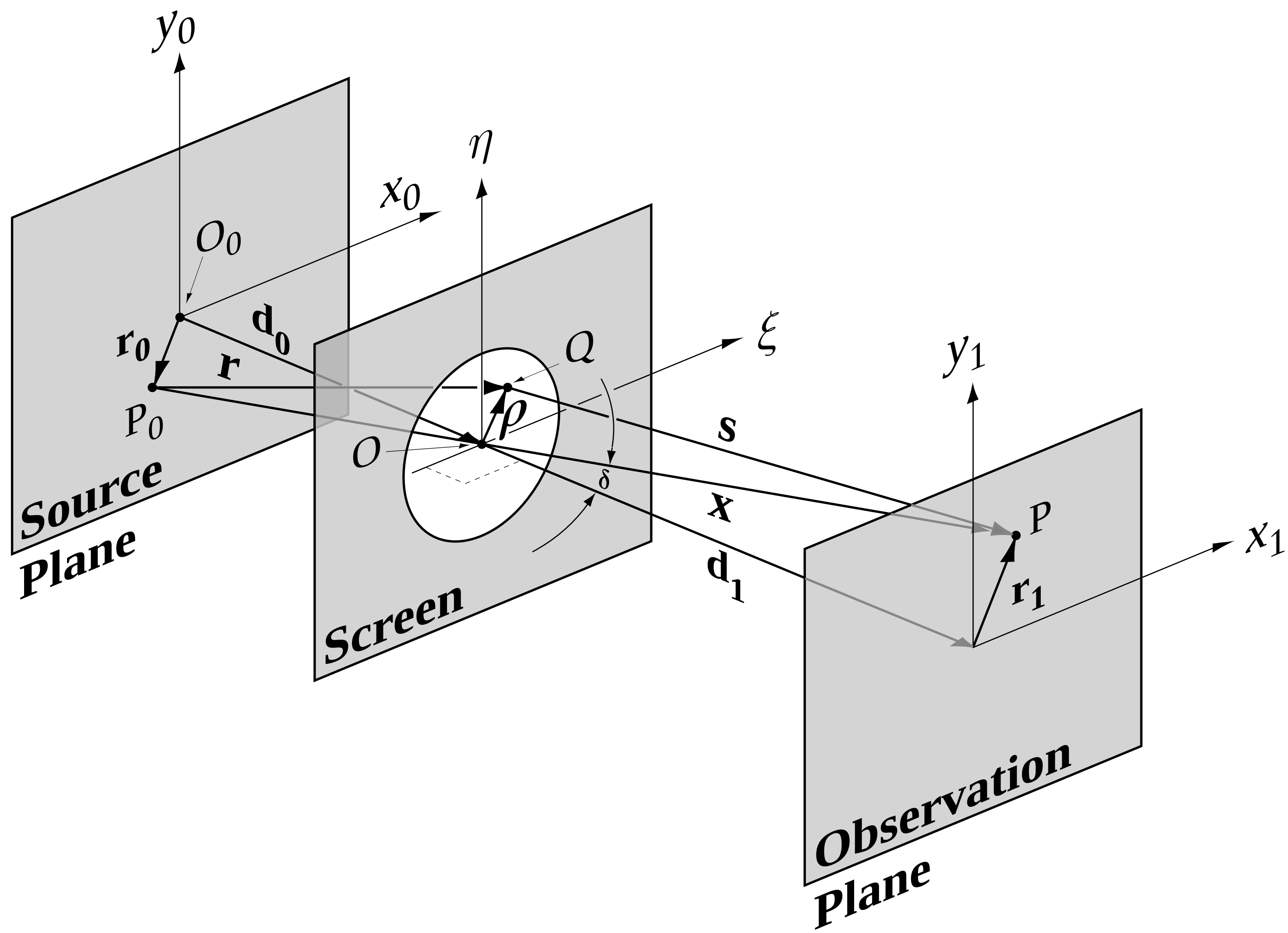
$$\nabla u(\mathbf{r}) = \hat{\mathbf{r}}\,A\frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right), \quad \nabla G(\mathbf{r}, \mathbf{x}) = -\hat{\mathbf{s}}\,A\frac{e^{iks}}{s}\left(ik - \frac{1}{s}\right)$$

$$\text{where} \quad \mathbf{s} := \mathbf{x} - \mathbf{r}$$

  - ▷ Result: The **Fresnel-Kirchhoff diffraction formula**,

$$\boxed{u(\mathbf{x}) = -\frac{ikA}{4\pi}\int_{\mathrm{A}} \frac{e^{ik(r+s)}}{r\,s}(\hat{\mathbf{n}}\cdot\hat{\mathbf{s}} + \hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\,dS}$$

COORDINATES FOR THE FRESNEL-KIRCHHOFF DIFFRACTION INTEGRAL



FRESNEL-KIRCHHOFF DIFFRACTION THEORY (9)

- The **Fresnel-Kirchhoff diffraction formula**

$$u(\mathbf{x}) = -\frac{ikA}{4\pi}\int_{\mathrm{A}} \frac{e^{ik(r+s)}}{rs}(\hat{\mathbf{n}}\cdot\hat{\mathbf{s}}+\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})\,dS$$

  ▷ is an approximate scalar solution of the Helmholtz equation, valid for wavelengths small compared to the aperture size

  ▷ is based on deep physical insight

  ▷ is accurate enough to give useful, quantitative results in optics

  BUT

  ▷ does not make $u \longrightarrow 0$ at large distances (as assumed in Kirchhoff's boundary conditions)

  ▷ is not the first term in any iterative solution of the boundary-value problem

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (10)

- Consistency of the **Fresnel-Kirchhoff diffraction formula**

$$u(\mathbf{x}) = -\frac{ikA}{4\pi} \int_{\mathrm{A}} \frac{e^{ik(r+s)}}{rs} (\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}})\, dS$$

with Huyghens' Principle

  - ▷ Expresses $u(\mathbf{x})$ as a superposition of spherical waves e^{iks}/s emanating from the wavefront e^{ikr}/r produced by a point source
  - ▷ The **inclination factor** is

$$I(\hat{\mathbf{r}}, \hat{\mathbf{s}}) = -(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}})$$

    - ∘ In the backward direction ($\hat{\mathbf{s}} = -\hat{\mathbf{r}}$),

$$I(\hat{\mathbf{r}}, -\hat{\mathbf{r}}) = 0$$

    - ∘ In the forward direction ($\hat{\mathbf{s}} = \hat{\mathbf{r}}$),

$$I(\hat{\mathbf{r}}, \hat{\mathbf{r}}) = -2\cos\delta$$

FRESNEL-KIRCHHOFF DIFFRACTION THEORY (11)

- Extensions of the Fresnel-Kirchhoff diffraction formula:
  - ▷ If the inclination factor is nearly constant over the aperture,

$$u(\mathbf{x}) \approx \frac{ikAI}{4\pi} \int_{\mathrm{A}} \frac{e^{ik(r+s)}}{rs}\, dS$$

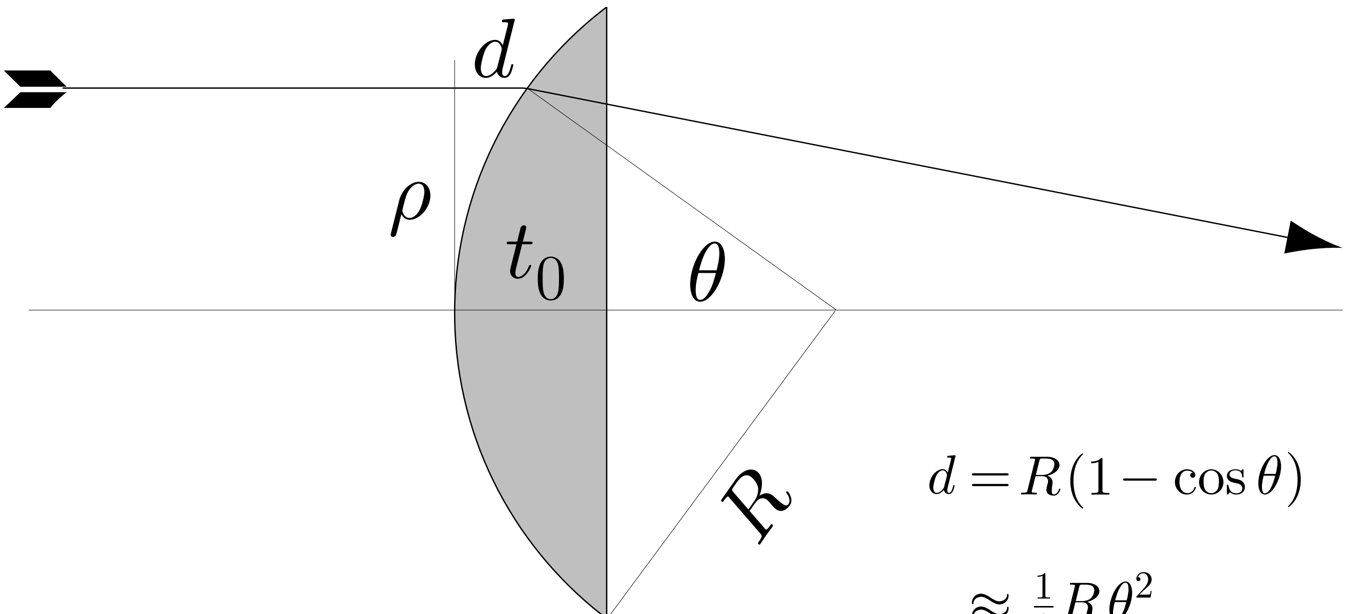
  - ▷ Replace the point-source incident wavefront e^{ikr}/r produced by a point source with a general wavefront $u_{\mathrm{inc}}(\mathbf{r})$ produced by an extended source (modeled as a collection of point sources):

$$u(\mathbf{x}) = \frac{ikI}{4\pi} \int_{\mathrm{A}} u_{\mathrm{inc}}(\mathbf{r}) \frac{e^{iks}}{s}\, dS$$

  - ▷ Characterize the aperture by a **transmission function** τ to model amplitude and/or phase changes due to lenses & diffraction gratings

$$\boxed{u(\mathbf{x}) = \frac{ikI}{4\pi} \int_{\mathrm{A}} \tau(\mathbf{r})\, u_{\mathrm{inc}}(\mathbf{r}) \frac{e^{iks}}{s}\, dS}$$

THIN LENS



$$d = R(1 - \cos\theta)$$

$$\approx \tfrac{1}{2} R\theta^2$$

$$\approx \frac{\rho^2}{2R}$$

Optical path in lens $\approx kn\left[t_0 - \dfrac{\rho^2}{2R}\right]$

TRANSMISSION FUNCTION OF A THIN LENS

- Transmission function of a thin lens (thickness $= t$):

$$\tau(\boldsymbol{\rho}) \approx |\tau(\boldsymbol{\rho})| e^{iknt} e^{-ik\rho^2/(2f)}$$

  - ▷ If the incident field is $u_{\rm inc}(\mathbf{r}) = e^{ikz}$, then the field transmitted by a plano-convex thin lens (neglecting reflections) is

$$u_{\rm tr}(\mathbf{r}) \approx \exp ik\left\{z + \frac{\rho^2}{2R} + n\left[t_0 - \frac{\rho^2}{2R}\right]\right\}$$

  - ▷ Therefore the plano-convex transmission function is

$$\tau_{\rm plano}(\boldsymbol{\rho}) \approx e^{iknt_0} e^{-ik(n-1)\rho^2/(2R)}$$

  - ▷ For a biconvex lens, with curvatures R_1 and $-R_2$,

$$\tau_{\rm biconvex}(\boldsymbol{\rho}) \approx e^{ikn(t_1+t_2)} e^{-ik(n-1)\rho^2/[(2R_1)^{-1}-(2R_2)^{-1}]}$$

  - ▷ Focal length f = net curvature: $$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

DIFFRACTION THEORY (1)

- Applications of the Fresnel-Kirchhoff diffraction formula

$$u(\mathbf{x}) = \frac{ikI}{4\pi} \int_{\mathrm{A}} \tau(\mathbf{r})\, u_{\mathrm{inc}}(\mathbf{r}) \frac{e^{iks}}{s}\, dS$$

  - ▷ Diffraction of light from a point source:

$$u_{\mathrm{inc}}(\mathbf{r}) = A\, \frac{e^{ikr}}{r}$$

    - ◦ Fraunhofer limit (plane waves)
    - ◦ Fresnel regime
  - ▷ Diffraction theory of image formation
    - ◦ Aberrations
    - ◦ Focal-plane filtering
    - ◦ Transfer functions
  - ▷ Propagation of the mutual coherence function

DIFFRACTION THEORY (2)

- Fresnel's approximation to a spherical wave:

$$\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1\cdot\boldsymbol{\rho}} e^{ik[\rho^2-(\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2]/(2r')}$$

  ▷ $\dfrac{e^{ikr'}}{r'}$ = spherical wave centered at P_0

  ▷ $e^{i\mathbf{k}_1\cdot\boldsymbol{\rho}}$ = plane wave with propagation constant $\mathbf{k}_1 := k\hat{\mathbf{r}}'$

  ▷ $e^{ik[\rho^2-(\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2]/(2r')}$ = lowest correction for wavefront curvature

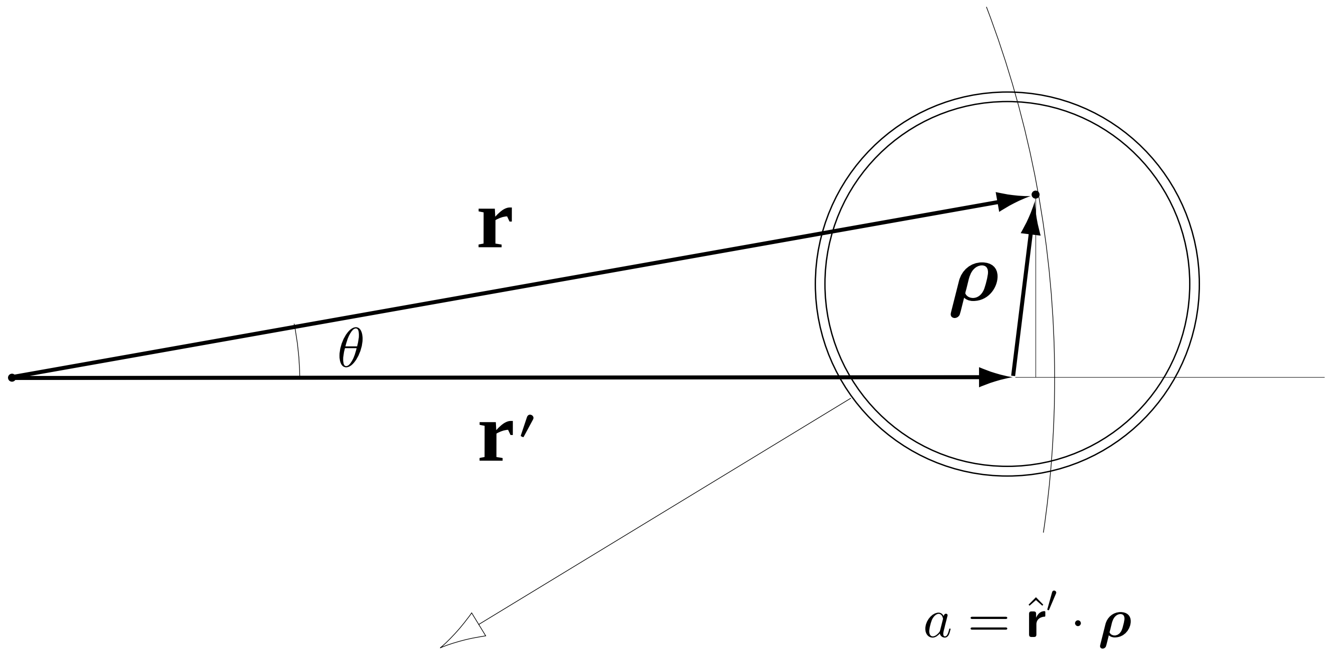
    ∘ This factor is approximately 1 when the Fresnel number is small:

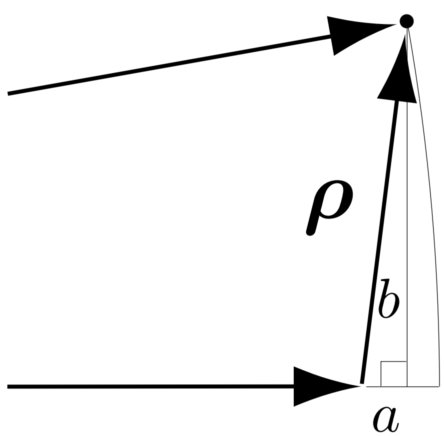
$$\frac{b^2}{r'\lambda} \ll 1$$

    where $b^2 := \rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2$

- Bottom line: You get a nearly plane wave if you move far away from a point source (but it's easier to use a lens!)

THE FRESNEL APPROXIMATION





$$a = \hat{\mathbf{r}}' \cdot \boldsymbol{\rho}$$

$$b^2 = \rho^2 - a^2 \;= \rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2$$

$$\begin{aligned} r &= \frac{r' + a}{\cos\theta} \\ &\approx \frac{r'}{\cos\theta} + a \approx r'(1 + \tfrac{1}{2}\theta^2) + a \\ &\approx r' + a + \frac{b^2}{2r'} \end{aligned}$$

FRESNEL APPROXIMATION DETAILS (1)

- Approximate r in terms of $\mathbf{r}'$ (fixed) and $\boldsymbol{\rho}$ (varies over aperture):
  - ▷ Vector addition:

$$\mathbf{r} = \mathbf{r}' + \boldsymbol{\rho}$$

$$\mathbf{r}^2 = \mathbf{r}'^2 + 2\mathbf{r}' \cdot \boldsymbol{\rho} + \boldsymbol{\rho}^2$$

$$r = r' \left[1 + 2\frac{\mathbf{r}'}{r'^2} \cdot \boldsymbol{\rho} + \left(\frac{\rho}{r'} \right)^2 \right]^{1/2}$$

  - ▷ Binomial expansion correct to order $(\rho/r')^2$:

$$r \approx r' + \frac{\mathbf{r}'}{r'} \cdot \boldsymbol{\rho} + \frac{\rho^2}{2r'} - \frac{(\mathbf{r}' \cdot \boldsymbol{\rho})^2}{2r'^3}$$

$$r \approx r' + \hat{\mathbf{r}}' \cdot \boldsymbol{\rho} + \frac{\rho^2}{2r'} - \frac{(\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'}$$

FRESNEL APPROXIMATION DETAILS (2)

- The **Fresnel approximation**, valid for small angles between $\mathbf{r}$ and r':

$$\text{denominator} = r \approx r'$$

$$\text{phase} = kr \approx kr' + \mathbf{k}_1 \cdot \boldsymbol{\rho} + k \left[\frac{\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2}{2r'} \right]$$

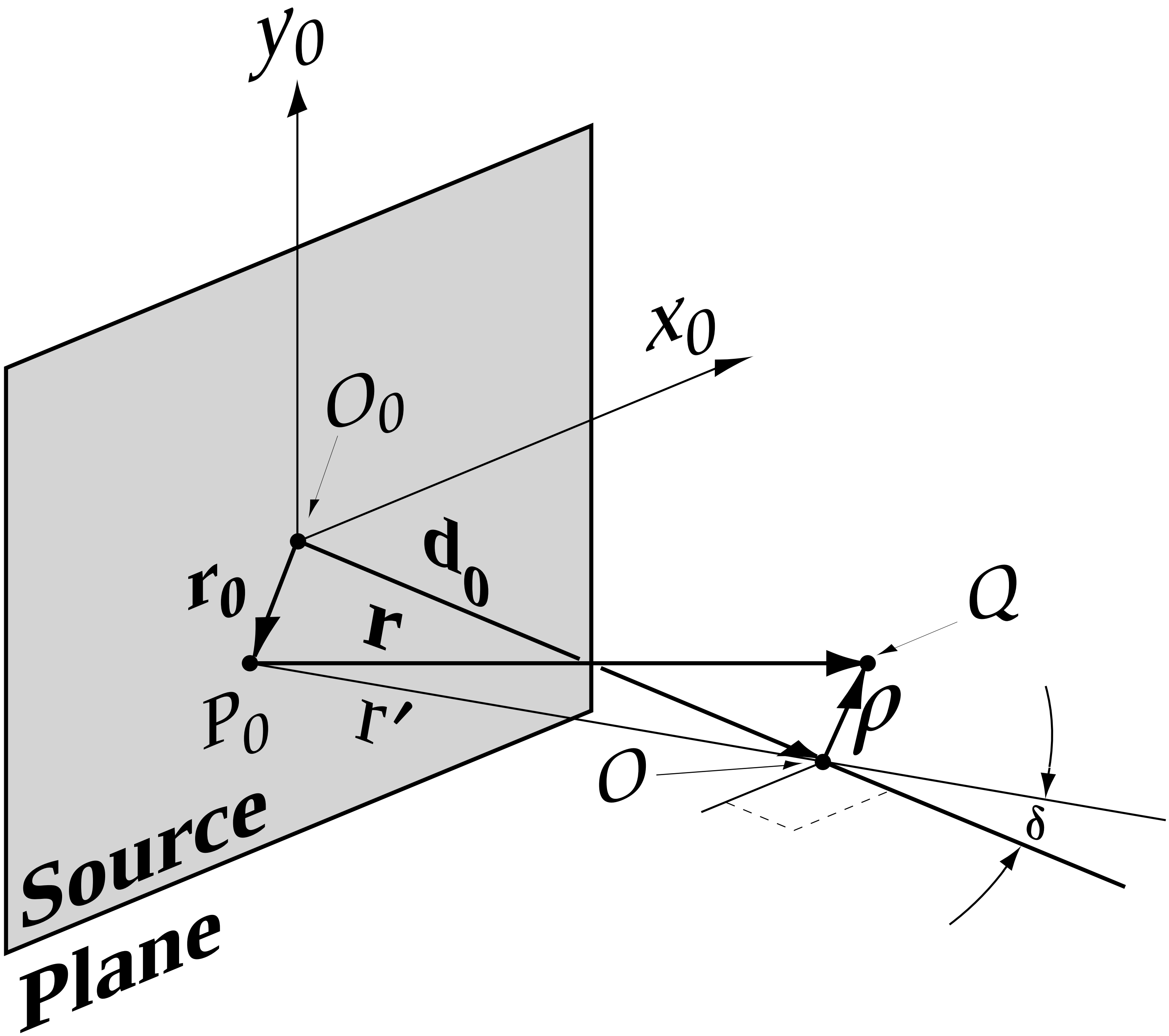
▷ $\dfrac{e^{ikr'}}{r'}$ = spherical wave centered at P_0

▷ $e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}}$ = plane wave with direction $\hat{\mathbf{r}}'$

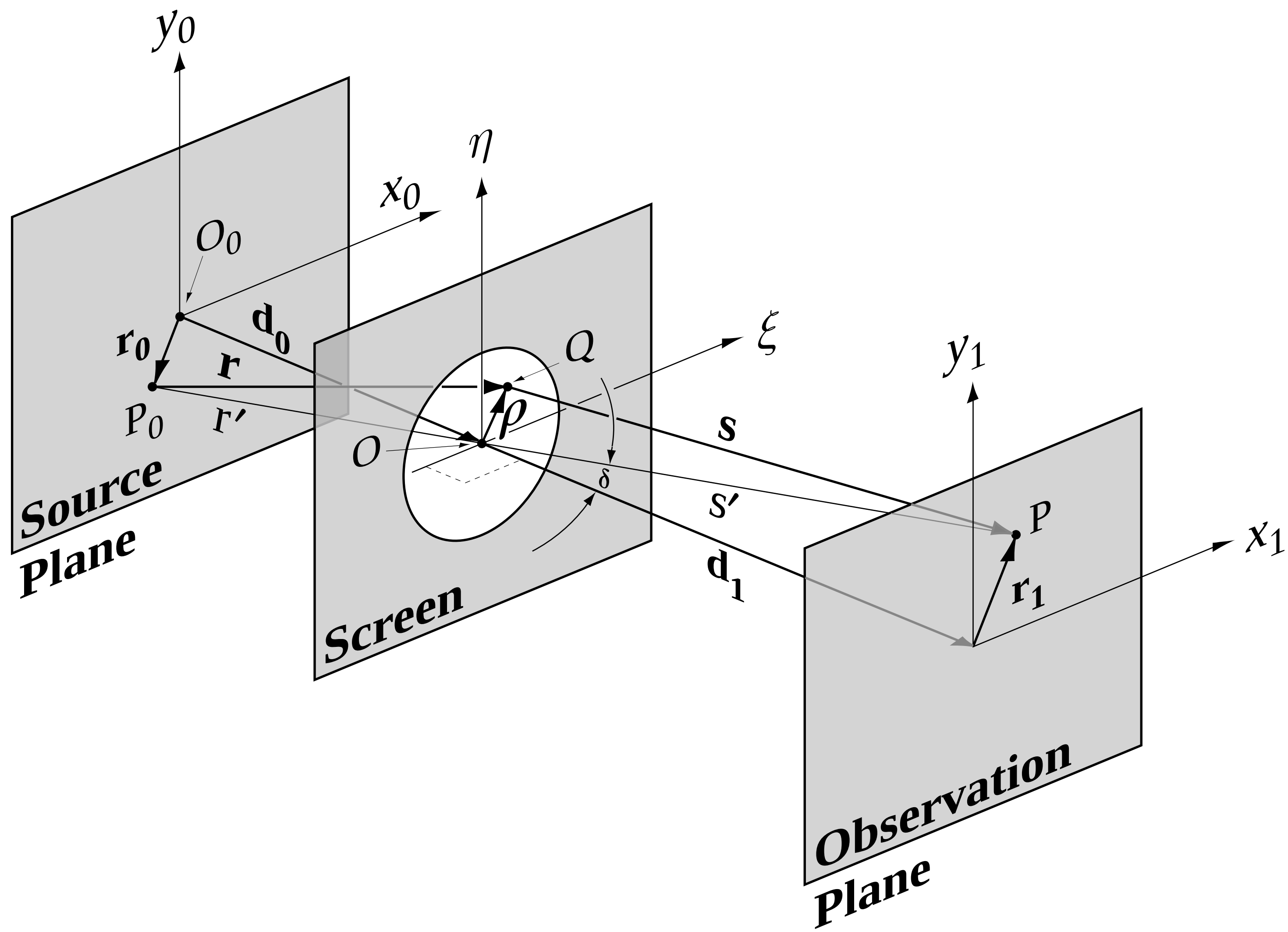
◦ Propagation constant: $\mathbf{k}_1 := k\hat{\mathbf{r}}'$

▷ $e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}$ = lowest-order approximation to the phase of a spherical wavefront

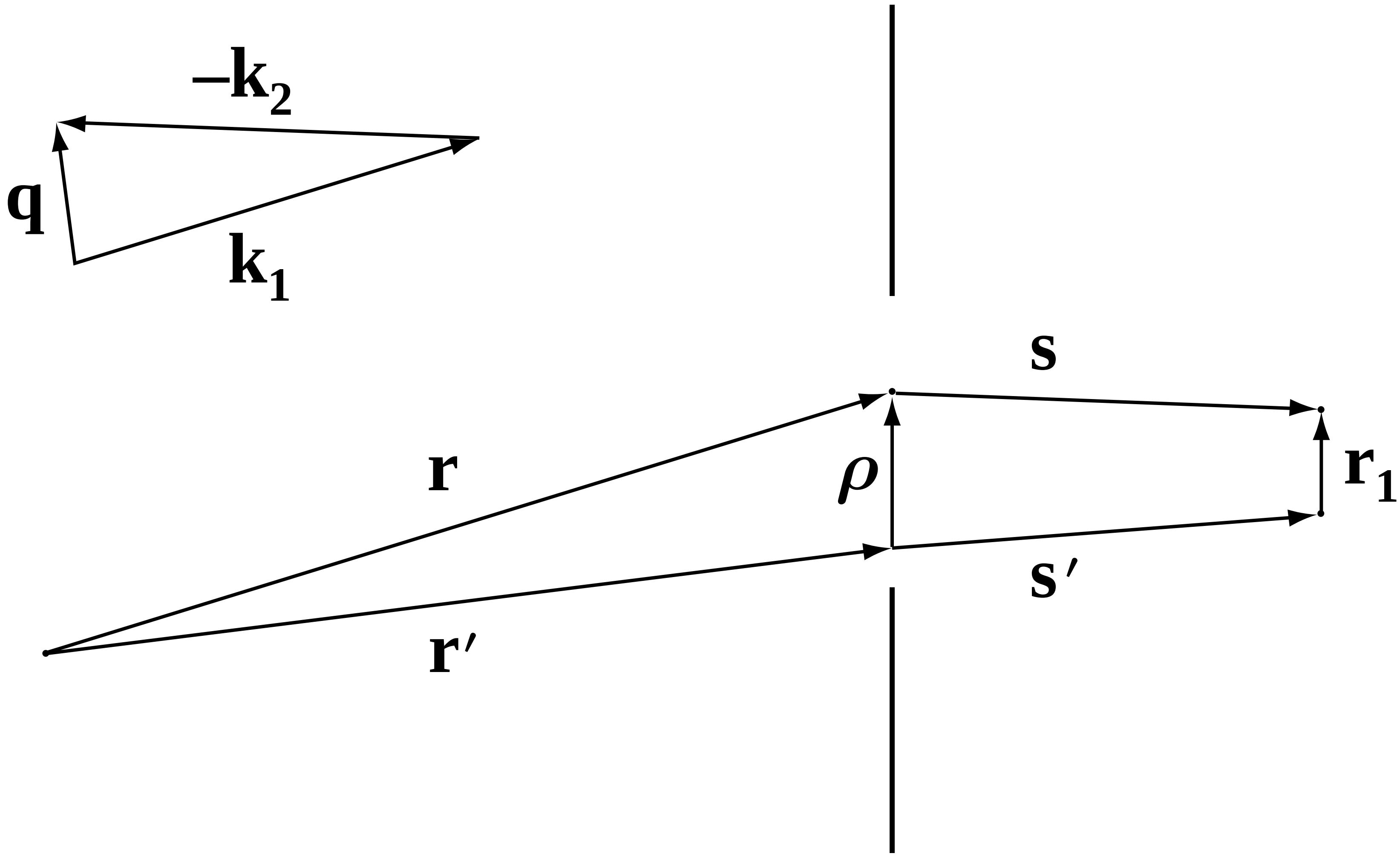
$$\boxed{\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1 \cdot \boldsymbol{\rho}} e^{ik[\rho^2 - (\hat{\mathbf{r}}' \cdot \boldsymbol{\rho})^2]/(2r')}}$$


y_0
x_0
O_0
$\mathbf{d_0}$
$\mathbf{r_0}$
$\mathbf{r}$
Q
P_0
r'
$\boldsymbol{\rho}$
O
δ
Source
Plane

COORDINATES FOR THE FRESNEL-KIRCHHOFF DIFFRACTION INTEGRAL



COORDINATES FOR THE DIFFRACTION INTEGRAL



DIFFRACTION THEORY (3)

- Diffraction pattern of a point source, part 1:
  - ▷ Fresnel's approximation applied to the spherical waves e^{ikr}/r and e^{iks}/s:

$$\frac{e^{ikr}}{r} \approx \frac{e^{ikr'}}{r'} e^{i\mathbf{k}_1\cdot\boldsymbol{\rho}} \exp\left\{ik\left[\frac{\rho^2 - (\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2}{2r'}\right]\right\}$$

$$\frac{e^{iks}}{s} \approx \frac{e^{iks'}}{s'} e^{i\mathbf{k}_2\cdot(\mathbf{r}_1-\boldsymbol{\rho})} \exp\left\{ik\left[\frac{(\mathbf{r}_1-\boldsymbol{\rho})^2 - [\hat{\mathbf{s}}'\cdot(\mathbf{r}_1-\boldsymbol{\rho})]^2}{2s'}\right]\right\}$$

  where the propagation vectors are

$$\mathbf{k}_1 := k\hat{\mathbf{r}}', \quad \mathbf{k}_2 := k\hat{\mathbf{s}}', \quad \mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2$$

  - ▷ Diffraction integral for a point source in Fresnel's approximation:

$$u(\mathbf{x}) = \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_{\mathrm{A}} \tau(\mathbf{r})\, e^{i\mathbf{q}\cdot\boldsymbol{\rho}} e^{ik\left[\frac{\rho^2-(\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2}{2r'} + \frac{(\mathbf{r}_1-\boldsymbol{\rho})^2-[\hat{\mathbf{s}}'\cdot(\mathbf{r}_1-\boldsymbol{\rho})]^2}{2s'}\right]} dS$$

DIFFRACTION THEORY (4)

- Diffraction pattern of a point source, part 2:
  - ▷ Diffraction integral for a point source in Fresnel's approximation:

$$u(\mathbf{x}) = \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_A \tau(\boldsymbol{\rho})\, e^{i\mathbf{q}\cdot\boldsymbol{\rho}} e^{ik\left[\frac{\rho^2-(\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2}{2r'}+\frac{(\mathbf{r}_1-\boldsymbol{\rho})^2-[\hat{\mathbf{s}}'\cdot(\mathbf{r}_1-\boldsymbol{\rho})]^2}{2s'}\right]}\, dS$$

    where $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2, \quad \mathbf{k}_1 := k\hat{\mathbf{r}}', \quad \mathbf{k}_2 := k\hat{\mathbf{s}}'$
  - ▷ $e^{i\mathbf{k}_2\cdot\mathbf{r}_1}$ = plane wave "carrier" incident at P
  - ▷ Integral over aperture = "signal"
    - ○ $e^{ik\left[\frac{\rho^2-(\hat{\mathbf{r}}'\cdot\boldsymbol{\rho})^2}{2r'}+\frac{(\mathbf{r}_1-\boldsymbol{\rho})^2-[\hat{\mathbf{s}}'\cdot(\mathbf{r}_1-\boldsymbol{\rho})]^2}{2s'}\right]}$ = wavefront curvature correction
    - ○ **Fresnel diffraction**: Wavefront curvature is an essential part of the physics
    - ○ **Fraunhofer diffraction**: No wavefront curvature correction needed

$$u(\mathbf{x}) = \int_A \tau(\boldsymbol{\rho})\, e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS$$

$$= \textbf{Fourier transform of } \tau$$

DIFFRACTION THEORY (5)

- Fraunhofer diffraction at a rectangular aperture:

$$u(\mathbf{x}) = \int_{\mathrm{A}} \tau(\boldsymbol{\rho})\, e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS$$

where $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2, \quad \mathbf{k}_1 := k\hat{\mathbf{r}}', \quad \mathbf{k}_2 := k\hat{\mathbf{s}}'$

▷ Aperture dimensions: $2a$ (in x) $\times$ $2b$ (in y)

▷ $\mathbf{q}$ is not necessarily in the plane of the aperture, but we can assume that it is for this case because its z-component is small

▷ Integral over the components of $\boldsymbol{\rho}$:

$$\begin{aligned} u(\mathbf{r}_1) &= \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_{-a}^{a}\int_{-b}^{b} e^{i(q_x\xi+q_y\eta)}\, d\xi\, d\eta \\ &= (\text{constant})[2\,\mathrm{sinc}\,(q_x a)][2\,\mathrm{sinc}\,(q_y b)] \end{aligned}$$

▷ Intensity:

$$\boxed{S(\mathbf{q}) = S(\mathbf{0})\,\mathrm{sinc}^2(q_x a)\,\mathrm{sinc}^2(q_y b)}$$

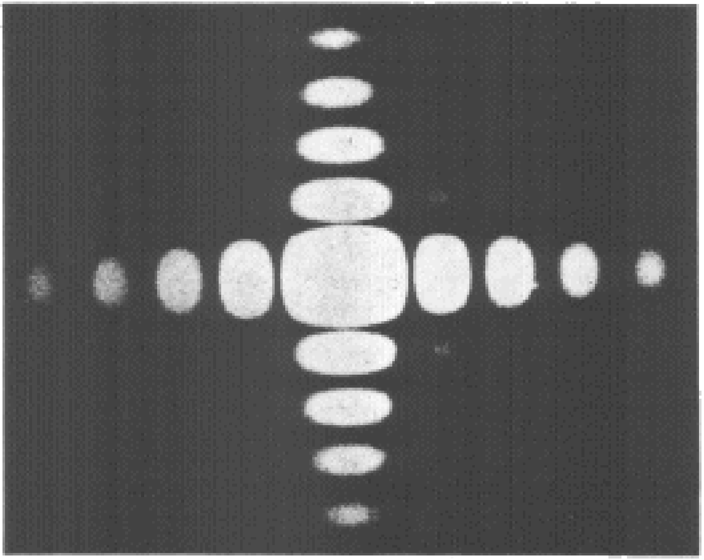


Fig. 8.10. FRAUNHOFER diffraction pattern of a rectangular aperture 8 mm × 7 mm, magnification 50×, mercury yellow light $\lambda = 5790$ Å. To show the existence of the weak secondary maxima the central portion was overexposed.

(After H. LIPSON, C. A. TAYLOR, and B. J. THOMPSON.)

DIFFRACTION THEORY (6)

- Fraunhofer diffraction at a circular aperture:

$$u(\mathbf{x}) = \int_{\mathrm{A}} \tau(\boldsymbol{\rho})\, e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS$$

where $\mathbf{q} := \mathbf{k}_1 - \mathbf{k}_2, \quad \mathbf{k}_1 := k\hat{\mathbf{r}}', \quad \mathbf{k}_2 := k\hat{\mathbf{s}}'$

▷ Aperture radius $= a$

▷ Integral over the components of $\boldsymbol{\rho}$:

$$u(\mathbf{q}) = \frac{ikIAe^{ik(r'+s')}}{4\pi r's'} e^{i\mathbf{k}_2\cdot\mathbf{r}_1} \int_0^{2\pi} \int_0^a e^{iq\rho\cos\phi}\, \rho\, d\rho\, d\phi$$

$$= (\text{constant})(\pi a^2) \left[\frac{2J_1(qa)}{qa}\right]$$

▷ Intensity:

$$S(\mathbf{q}) = S(\mathbf{0}) \left[\frac{2J_1(qa)}{qa}\right]^2$$

(the **Airy pattern**)

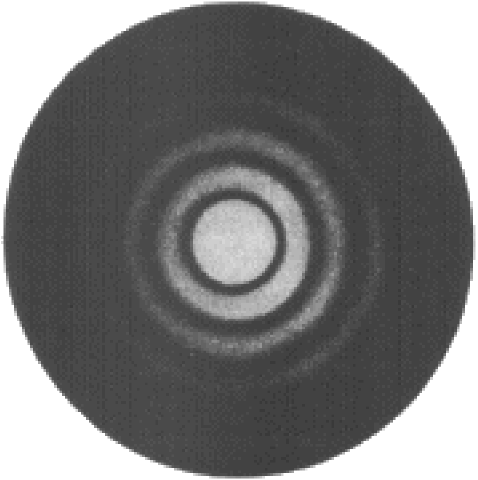


Fig. 8.12. FRAUNHOFER diffraction pattern of a circular aperture (the AIRY pattern) 6 mm in diameter, magnification 50×, mercury yellow light $\lambda = 5790$ Å. To show the existence of the weak subsidiary maxima, the central portion was overexposed.

(After H. LIPSON, C. A. TAYLOR, and B. J. THOMPSON.)

DIFFRACTION THEORY (7)

- Dimensions of central feature:
  - ▷ Rectangular aperture, $2a \times 2b$
    - ∘ For a small diffraction angle, $q_x \approx k\dfrac{\xi}{s'}, \quad q_y \approx k\dfrac{\eta}{s'}$
    - ∘ First zero of sinc function is at $x = \pi$
    - ∘ At the first zero in ξ, $\dfrac{2\pi}{\lambda}\dfrac{\xi_0}{s'}a = \pi$

$$\boxed{\xi_0 = \frac{\lambda s'}{2a}, \quad \eta_0 = \frac{\lambda s'}{2b}}$$

  - ▷ Circular aperture, radius a
    - ∘ For a small diffraction angle θ, $q \approx k\theta$
    - ∘ First zero of J_1 is $j_{1,1} = 3.832\cdots = 1.220\pi$
    - ∘ At the first zero, $qa \approx 2\pi\theta_0 a/\lambda \approx 1.220\pi$

$$\boxed{\theta_0 \approx 0.610\frac{\lambda}{a} \approx 1.220\frac{\lambda}{d}}$$

DIFFRACTION THEORY (8)

- Fraunhofer diffraction by N randomly arranged identical apertures:
  - ▷ Aperture A_m is displaced from aperture A_1 by a random vector $\mathbf{a}_m$
  - ▷ Amplitude of diffracted field is proportional to

$$\int_A e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS = \sum_{m=1}^{N} e^{i\mathbf{q}\cdot\mathbf{a}_m} \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS$$

  - ▷ Intensity of diffracted field is

$$S(\mathbf{q}) = (\text{const.}) \left| \sum_{m=1}^{N} e^{i\mathbf{q}\cdot\mathbf{a}_m} \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS \right|^2$$

$$= (\text{const.}) \sum_{m=1}^{N} \sum_{n=1}^{N} e^{i\mathbf{q}\cdot(\mathbf{a}_m - \mathbf{a}_n)} \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS \right|^2 = N \times (\text{const.}) \times \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS \right|^2$$

$$\left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS \right|^2 \propto \text{diffraction pattern of a single aperture}$$

  - ▷ Intensities add, not amplitudes, when the arrangement is random

DIFFRACTION THEORY (9)

- Fraunhofer diffraction by periodically arranged identical apertures, part 1:
  - ▷ The same unit cell is translated through integer multiples of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (not necessarily orthogonal!)
  - ▷ Amplitude of diffracted field is proportional to

$$\int_A e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS = \sum_{m_1=-M_1}^{M_1}\sum_{m_2=-M_2}^{M_2} e^{i\mathbf{q}\cdot(m_1\mathbf{a}_1+m_2\mathbf{a}_2)}\int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS$$

$$= g_1(\mathbf{q}\cdot\mathbf{a}_1)g_2(\mathbf{q}\cdot\mathbf{a}_2)\int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,dS$$

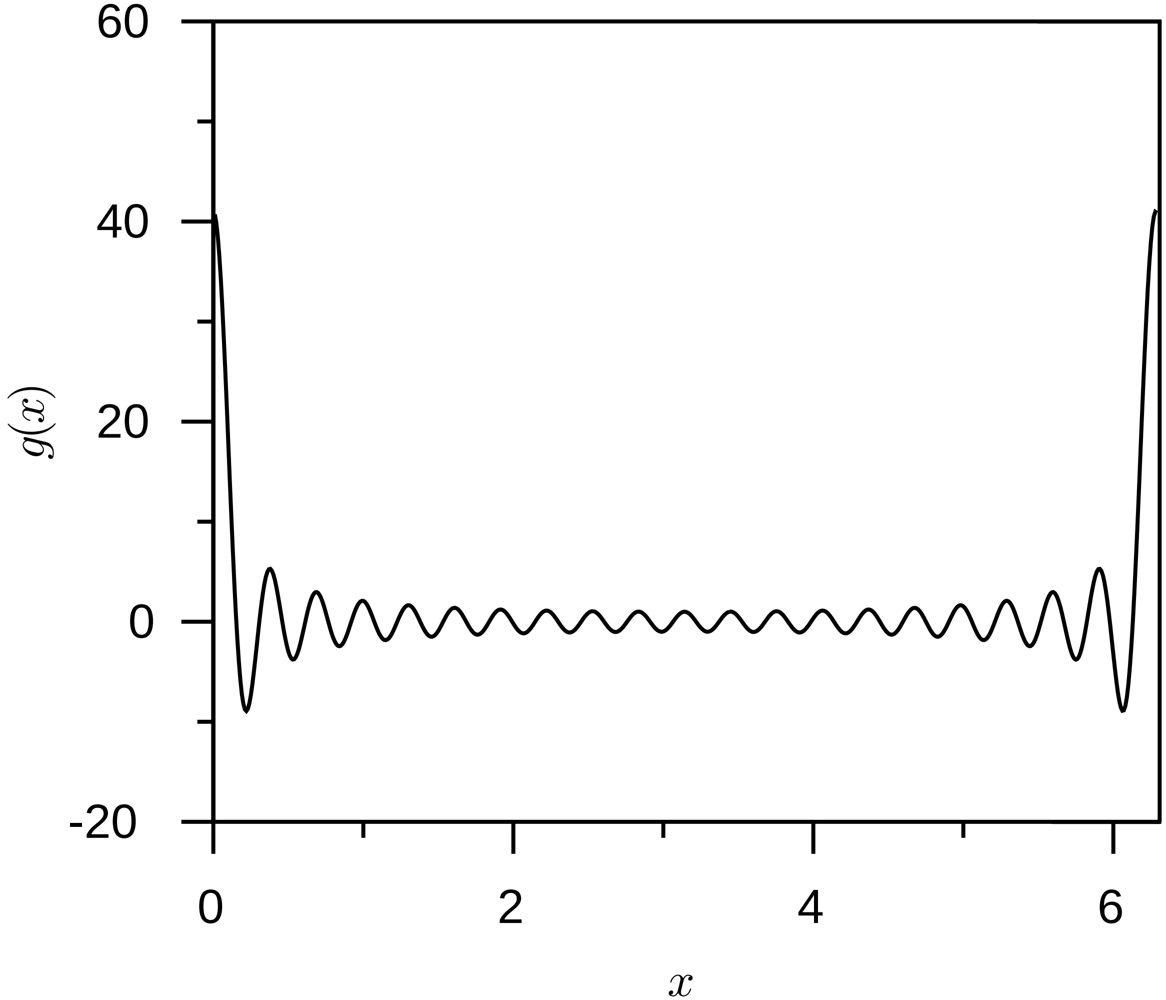
  - ▷ Grating functions:

$$g_j(\mathbf{q}\cdot\mathbf{a}_j) = \sum_{m_j=-M_j}^{M_j} e^{im_j\mathbf{q}\cdot\mathbf{a}_j}$$

$$= \frac{\sin\left[(M_j+\frac{1}{2})\mathbf{q}\cdot\mathbf{a}_j\right]}{\sin\frac{1}{2}\mathbf{q}\cdot\mathbf{a}_j}$$

THE GRATING FUNCTION

$$g(x) = \frac{\sin\left[(M+\frac{1}{2})x\right]}{\sin(\frac{1}{2}x)} \text{ for } M = 20$$



DIFFRACTION THEORY (10)

- Fraunhofer diffraction by periodically arranged identical apertures, part 2:
  - ▷ Properties of the grating function

$$g(\mathbf{q}\cdot\mathbf{a}) = \frac{\sin\left[(M+\frac{1}{2})\mathbf{q}\cdot\mathbf{a}\right]}{\sin\frac{1}{2}\mathbf{q}\cdot\mathbf{a}}$$

- ∘ Peak value of g is $2M+1$ at

$$\mathbf{q}\cdot\mathbf{a} = 2n\pi$$

- ∘ Half-width of central peak $= \dfrac{\pi}{M+\frac{1}{2}}$
- ∘ Amplitude of first side lobe is $-0.212\times$ amplitude of central peak

DIFFRACTION THEORY (11)

- Fraunhofer diffraction by periodically arranged identical apertures, part 3:
  - ▷ Intensity of diffracted field is proportional to

$$\left| \int_A e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS \right|^2 = |g_1(\mathbf{q}\cdot\mathbf{a}_1)|^2\, |g_2(\mathbf{q}\cdot\mathbf{a}_2)|^2 \left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS \right|^2$$

  - ∘ The grating functions are sharply peaked at values of $\mathbf{q}$ such that

$$\mathbf{q}\cdot\mathbf{a}_1 = 2n_1\pi \quad \text{and} \quad \mathbf{q}\cdot\mathbf{a}_2 = 2n_2\pi$$

    - ⋄ These conditions imply that $\mathbf{q}$ is 2π times a reciprocal lattice vector:

$$\mathbf{q} = 2\pi(n_1\mathbf{b}_1 + n_2\mathbf{b}_2)$$

  - ∘ Peaks are modulated by the diffraction pattern of a single aperture, $\left| \int_{A_1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\, dS \right|^2$

2-D RECIPROCAL LATTICE (1)

- The 2-d direct lattice spanned by non-parallel vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ consists of the points

$$\mathbf{r}_{m_1,m_2} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2$$

- The **reciprocal lattice** to the $\mathbf{a}_1 - \mathbf{a}_2$ lattice consists of vectors

$$\mathbf{k}_{n_1,n_2} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2$$

where

$$\mathbf{b}_i \cdot \mathbf{a}_j = \delta_{i,j}$$

▷ The **reciprocal lattice basis** is

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \hat{\mathbf{z}}}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}, \quad \mathbf{b}_2 = \frac{\hat{\mathbf{z}} \times \mathbf{a}_1}{\hat{\mathbf{z}} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}$$

Hexagonal lattice basis: $\mathbf{a}_1 = \hat{\mathbf{x}}, \quad \mathbf{a}_2 = \frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}}$

Reciprocal lattice basis: $\mathbf{b}_1 = \hat{\mathbf{x}} - \frac{1}{\sqrt{3}}\hat{\mathbf{y}}, \quad \mathbf{b}_2 = \frac{2}{\sqrt{3}}\hat{\mathbf{y}}$

DIFFRACTION THEORY (12)

- Calculation of the field in the image plane, part 1:
  - ▷ Assume normal incidence
    - ∘ Inclination factor

$$I = -(\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} + \hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) = -2$$

    - ∘ Diffraction integral assuming a point source and a thin lens with focal length f:

$$\begin{aligned} u(\mathbf{r}_1) &= -\frac{iA}{\lambda} \int_{\mathrm{A}} \frac{e^{ik(r+s)}}{rs} \, \tau(\boldsymbol{\rho}) \, dS \\ &\approx -\frac{iA}{\lambda} \int_{\mathrm{A}} \frac{e^{ik(r+s)}}{rs} \, e^{-ik\frac{\rho^2}{2f}} \, \alpha(\boldsymbol{\rho}) \, d^2\rho \end{aligned}$$

      - ◇ τ = transmission function of lens and aperture
      - ◇ α = transmission function of aperture alone

DIFFRACTION THEORY (13)

- Calculation of the field in the image plane, part 2:
  - ▷ Diffraction integral assuming a point source and a thin lens with focal length f:

$$u(\mathbf{r}_1) \approx -\frac{iA}{\lambda} \int_A \frac{e^{ik(r+s)}}{rs}\, e^{-ik\frac{\rho^2}{2f}}\, \alpha(\boldsymbol{\rho})\, d^2\rho$$

    - ◦ The sum of the exponents is (in the Fresnel approximation)

$$ik\left[r + s - \frac{\rho^2}{2f}\right] \approx ik\left[r' + s' - \frac{\rho^2}{2f} + \frac{(\mathbf{r}_0 - \boldsymbol{\rho})^2}{2r'} + \frac{(\boldsymbol{\rho} - \mathbf{r}_1)^2}{2s'}\right]$$

$$= ik\left[r' + s' + \frac{\rho^2}{2}\left(-\frac{1}{f} + \frac{1}{r'} + \frac{1}{s'}\right) + \frac{r_0^2}{2r'} + \frac{r_1^2}{2s'} - \boldsymbol{\rho}\cdot\left(\frac{\mathbf{r}_0}{r'} + \frac{\mathbf{r}_1}{s'}\right)\right]$$

DIFFRACTION THEORY (14)

- Calculation of the field in the image plane, part 3:
  - ▷ At the Gaussian focal point of a thin lens,

$$-\frac{1}{f}+\frac{1}{r'}+\frac{1}{s'}=0$$

    - ◦ This makes the coefficient of ρ^2 vanish in the exponent
    - ◦ The diffraction integral reduces to a phase factor times the Fourier transform of the **pupil function** α:

$$u(\mathbf{r}_1)\approx-\frac{iA}{\lambda r's'}e^{ik\left[\left(\frac{r_0^2}{2r'}+\frac{r_1^2}{2s'}\right)+r'+s'\right]}\int_{\mathrm{A}}\alpha(\boldsymbol{\rho})\,e^{-ik\boldsymbol{\rho}\cdot\left(\frac{\mathbf{r}_0}{r'}+\frac{\mathbf{r}_1}{s'}\right)}\,d^2\rho$$

    - ◦ The image is spread out by diffraction
    - ◦ The peak amplitude occurs at the stationary-phase point:

$$\frac{\mathbf{r}_0}{r'}+\frac{\mathbf{r}_1}{s'}=\mathbf{0}\quad\Rightarrow\quad\mathbf{r}_1=-\frac{s'}{r'}\mathbf{r}_0\quad\text{(Gaussian image displacement)}$$

DIFFRACTION THEORY (15)

- Imaging by a thin lens of an extended, coherently illuminated object:
  - ▷ Superpose the fields due to a distribution ψ of point sources:

$$u(\mathbf{r}_1) \approx -\frac{i}{\lambda} \int_{\mathrm{S}} \int_{\mathrm{A}} \psi(\mathbf{r}_0) \, \frac{e^{ikr}}{r} \, \alpha(\boldsymbol{\rho}) \, e^{-ik\frac{\rho^2}{2f}} \, \frac{e^{iks}}{s} \, d^2\rho \, d^2 r_0$$

    - ◦ Instead of performing two 2-d integrations simultaneously, we can evaluate a diffraction integral to go from the source to the lens and another to go from the lens to the observation plane
    - ◦ Field incident at lens:

$$u_{\mathrm{inc}}(\boldsymbol{\rho}) \approx -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \int_{\mathrm{S}} \psi(\mathbf{r}_0) \, e^{\frac{ik(\boldsymbol{\rho}-\mathbf{r}_0)^2}{2r'}} \, d^2 r_0$$

$$\approx -\frac{i}{\lambda} \frac{e^{ikr'}}{r'} \int_{\mathrm{S}} \psi(\mathbf{r}_0) \, F_{k/r'}(\boldsymbol{\rho} - \mathbf{r}_0) \, d^2 r_0$$

      - ◇ u_{inc} is a convolution of the source distribution ψ with the Fresnel function $F_{k/r'}$, where

$$F_\alpha(\mathbf{r}) := e^{i\alpha r^2}$$

THE FRESNEL FUNCTION (1)

- The **Fresnel function** is

$$F_\alpha(\mathbf{r}) := e^{\frac{1}{2}i\alpha r^2}$$

where $\mathbf{r}$ lies in the $x - y$ plane

▷ The 2-d Fourier transform of F_α is

$$\begin{aligned}\tilde{F}_\alpha(\mathbf{q}) &= \int e^{-i\mathbf{q}\cdot\mathbf{r}}\, F_\alpha(\mathbf{r})\, d^2r \\ &= \frac{2i\pi}{\alpha} F_{-1/\alpha}(\mathbf{q}) \\ &= \frac{2i\pi}{\alpha} e^{-i\frac{q^2}{2\alpha}}\end{aligned}$$

DIFFRACTION THEORY (16)

- Imaging by a thin lens of an extended, coherently illuminated object, part 2:
  - ▷ The field incident on the lens, $u_{\rm inc}$, is a convolution of the source distribution ψ with the Fresnel function $F_{k/r'}$:

$$u_{\rm inc}(\boldsymbol{\rho}) = -\frac{i}{\lambda}\frac{e^{ikr'}}{r'}\int_{\rm S}\psi(\mathbf{r}_0)\,F_{k/r'}(\boldsymbol{\rho}-\mathbf{r}_0)\,d^2r_0$$

$$= -\frac{i}{\lambda}\frac{e^{ikr'}}{r'}\frac{1}{(2\pi)^2}\int\tilde{\psi}(\mathbf{q})\,\tilde{F}_{k/r'}(\mathbf{q})\,e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\,d^2q$$

  - ▷ Diffraction integral for the field in the observation plane:

$$u(\mathbf{r}_1) = \left(-\frac{i}{\lambda}\right)^2\frac{e^{ik(r'+s')}}{(2\pi)^2r's'}\int\int_{\rm A}\tilde{\psi}(\mathbf{q})\,\tilde{F}_{k/r'}(\mathbf{q})\,\alpha(\boldsymbol{\rho})\,e^{E(\mathbf{q},\boldsymbol{\rho},\mathbf{r}_1)}\,d^2\rho\,d^2q$$

    - ∘ The sum of the exponents is (in the Fresnel approximation)

$$E(\mathbf{q},\boldsymbol{\rho},\mathbf{r}_1) = ik\left[-\frac{\rho^2}{2f}+\frac{\rho^2}{2s'}+\frac{r_1^2}{2s'}\right]+i\left(\mathbf{q}-k\frac{\mathbf{r}_1}{s'}\right)\cdot\boldsymbol{\rho}$$

DIFFRACTION THEORY (17)

- Imaging by a thin lens of an extended, coherently illuminated object, part 3:
  - ▷ In the back focal plane
  $$s' = f$$
  the sum of the exponents does not depend on ρ^2:
  $$E(\mathbf{q}, \boldsymbol{\rho}, \mathbf{r}_1) = ik\frac{r_1^2}{2s'} + i\mathbf{q}' \cdot \boldsymbol{\rho}$$
  where
  $$\mathbf{r}_1' := \mathbf{r}_1|_{s'=f} \quad \text{and} \quad \mathbf{q}' := \mathbf{q} - k\frac{\mathbf{r}_1'}{s'}$$
  - ▷ Diffraction integral for the field in the back focal plane:
  $$u(\mathbf{r}_1') = \left(-\frac{i}{\lambda}\right)^2 \frac{e^{ik(r'+s')}}{(2\pi)^2 r' s'} e^{ik\frac{{r_1'}^2}{2f}} \int \tilde{\psi}(\mathbf{q})\, \tilde{F}_{k/r'}(\mathbf{q})\, \tilde{\alpha}(\mathbf{q}')\, d^2q$$

DIFFRACTION THEORY (18)

- Imaging by a thin lens of an extended, coherently illuminated object, part 4:
  - ▷ Diffraction integral for the field in the back focal plane:

$$u(\mathbf{r}'_1) = \left(-\frac{i}{\lambda}\right)^2 \frac{e^{ik(r'+s')}}{(2\pi)^2 r' s'} e^{ik\frac{{r'_1}^2}{2f}} \int \tilde{\psi}(\mathbf{q})\, \tilde{F}_{k/r'}(\mathbf{q})\, \tilde{\alpha}(\mathbf{q}')\, d^2q$$

  - ▷ The Fourier transform of the pupil function, $\tilde{\alpha}$, is very sharply peaked at $\mathbf{q}' = \mathbf{0}$ (for a circular pupil, $\tilde{\alpha}$ is the Airy disk)
    - ∘ For an infinite aperture, $\tilde{\alpha}(\mathbf{q}') = (2\pi)^2 \delta(\mathbf{q}') \quad \Rightarrow \quad \mathbf{q} = k\frac{\mathbf{r}'_1}{f}$
    - ∘ In this limit, and with the Fraunhofer approximation,

$$\boxed{u(\mathbf{r}'_1) \approx -\frac{i}{\lambda f}\tilde{\psi}\left(k\frac{\mathbf{r}'_1}{f}\right)}$$

    - ∘ **The back focal plane contains the Fourier transform of a coherently illuminated object**

DIFFRACTION THEORY (19)

- Imaging by a thin lens of an extended, coherently illuminated object, part 5: **Focal plane filtering**
  - ▷ Concept: Alter the image by (intentionally or unintentionally) filtering the Fourier transform in the focal plane
  - ▷ Propagate the field from the back focal plane to the image plane using yet another diffraction integral!
  - ▷ Field in the image plane:

$$u(\mathbf{r}_1) = -\frac{i}{\lambda} \int_{\mathrm{F}} \frac{e^{iku}}{u} \, \sigma(\mathbf{r}_1') \, u(\mathbf{r}_1') \, d^2 r_1'$$

    - ∘ Focal-plane filter transmission function $= \sigma$
    - ∘ Vector from point in focal plane to point in image plane:

$$\mathbf{u} = \mathbf{s}'' + \mathbf{r}_1 - \mathbf{r}_1'$$

DIFFRACTION THEORY (20)

- Imaging by a thin lens of an extended, coherently illuminated object, part 6
  - ▷ Propagate the field from the back focal plane to the image plane
    - ∘ Assume an infinite aperture, $\tilde{\alpha}(\mathbf{q}') = (2\pi)^2\delta(\mathbf{q}')$:

$$u(\mathbf{r}_1') \approx -\frac{i}{\lambda f}\, e^{\frac{ikr_1'^2}{2}\left(\frac{1}{f}-\frac{r'}{f^2}\right)}\, \tilde{\psi}\left(k\frac{\mathbf{r}_1'}{s'}\right)$$

    - ∘ Sum of exponents in the diffraction integral:

$$\frac{ikr_1'^2}{2}\left(\frac{1}{f}-\frac{r'}{f^2}+\frac{1}{s''}\right)$$

    - ∘ Use Newton's equation for the focus-to-image distance r_I:

$$\frac{1}{f}-\frac{r'}{f^2} = \frac{f-r'}{f^2} = \frac{1}{f-r_I}$$

    - ∘ All quadratic phase terms cancel in the image plane, $f - r_I = -s''$

DIFFRACTION THEORY (21)

- Imaging by a thin lens of an extended, coherently illuminated object, part 7
  - ▷ Field amplitude in the image plane:

$$u(\mathbf{r}_1') \approx -\left(\frac{i}{\lambda}\right)^2 \frac{1}{(2\pi)^2 s'' f} e^{i\frac{kr_1^2}{2s''}} \int_{\mathrm{F}} \sigma(\mathbf{r}_1')\, \tilde{\psi}\left(k\frac{\mathbf{r}_1'}{f}\right) e^{-ik\frac{\mathbf{r}_1'\cdot\mathbf{r}_1}{s''}}\, d^2r_1'$$

    - ◦ **The image-plane field is proportional to the Fourier transform of the product of the focal-plane filtering function σ and the Fourier transform of the original field**