Time-Dependent Circuits Review

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1 INTRODUCTION

When you took introductory electromagnetism you studied circuits where the current changed with time, but you were so terrified of differential equations that you remember these circuits as being much harder to understand than they really are. Now that you have had differential equations it is time to review these circuits again. In the process of applying what you know about differential equations to circuits, you will also come to better understand differential equations.

The basic circuit analysis tools are the loop rules, so let's begin by reviewing them. The loop rules for circuits come from one of Maxwell's equations (Faraday's Law) in integral form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi_B}{\partial t} \quad . \tag{1}$$

The voltage differences across capacitors, resistors, and batteries come in through the line integral on the left side of this equation while the voltage differences due to inductors and generators come in through the time derivative of the magnetic flux on the right. We simplify this rather complicated situation by combining the inductive effects with the $\int \mathbf{E} \cdot d\mathbf{l}$ ones in the form

$$\sum \Delta V_i = 0 \quad , \tag{2}$$

with appropriate rules for the voltage changes ΔV_i across each circuit element. To apply these rules to a circuit, identify a closed path around some appropriate part of the circuit, choose a direction to follow around this path, and apply the following rules as you follow the wires and encounter circuit elements.

1. The voltage difference ΔV along a connecting wire is taken to be zero. This is actually not correct because wires have resistance, but it is usually safe to assume that the resistive voltage differences along the wires are much smaller than those across the circuit elements.

2. When you traverse a battery, or some other source of emf, in the direction that it tries to pump current, write down

$$\Delta V = \varepsilon \quad . \tag{3}$$

If you traverse it opposite to its pumping direction write

$$\Delta V = -\varepsilon \quad . \tag{4}$$

Don't pay any attention to the direction of the current through the emf; only its pumping direction matters.

3. If you traverse a capacitor from its negative side to its positive side write

$$\Delta V = Q/C \quad ; \tag{5}$$

if you traverse it from plus to minus write

$$\Delta V = -Q/C \quad ; \tag{6}$$

The charge Q is the charge on the positive plate of the capacitor.

4. If you traverse a resistor in the direction of the current flow through it write

$$\Delta V = -IR \quad ; \tag{7}$$

if you traverse it opposite to the current write

$$\Delta V = IR \quad . \tag{8}$$

And what do you do if you don't know the direction of the current flow? Make a guess, apply these rules, and solve the equations. If you guessed wrong the current will come out negative.

5. If you traverse an inductor in the direction of the current flow through it write

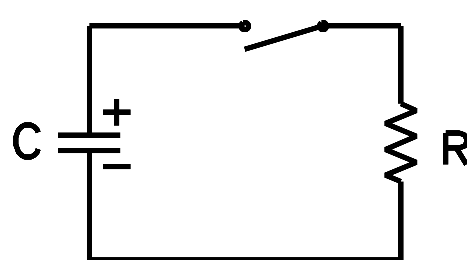
$$\Delta V = -L\frac{dI}{dt} \quad ; \tag{9}$$

if you traverse it opposite to the current write

$$\Delta V = L\frac{dI}{dt} \quad . \tag{10}$$

OK; that's all there is to it. Now let's study some circuits.

2 RC CIRCUITS



2.1 Discharging

Consider a charged capacitor C connected through a switch to a resistor R, as shown above. What happens when the switch is closed? Well, the capacitor dumps its charge by driving current through the resistor. To analyze this situation mathematically we first have to make two arbitrary, but important, choices. First we have to choose a path, including its direction, and second we have to choose the direction of positive current. In this problem let's follow a clockwise path around the circuit and let's let the direction of positive current I also be in the clockwise direction. Applying the loop rules at some instant of time after the switch has been closed gives the equation

$$\frac{Q}{C} - IR = 0 \quad , \tag{11}$$

where both Q and I are functions of time because as the capacitor discharges its voltage decreases. But this is only one equation for the two time-dependent functions $Q(t)$ and $I(t)$, so we need another one. The other equation is simply the definition of the current in terms of charge flow. I suggest that you always write it this way

$$I(t) = \pm \frac{dQ}{dt} \tag{12}$$

to remind yourself that both I and Q are signed quantities and that if you're not careful you will make a sign mistake. To choose the correct sign, look at the circuit again. In particular, look at the positive plate of the capacitor. We chose positive current to be clockwise, so if I is positive then it is draining charge away from this positive capacitor plate. This means that the charge Q on this plate is decreasing in time, so $dQ/dt < 0$. Equation (12) can only be valid if both sides have the same sign, so we choose

$$I(t) = -\frac{dQ}{dt} \tag{13}$$

With this choice for the connection between current and charge Eq. (11) can be written

$$\frac{dQ}{dt} + \frac{Q}{RC} = 0 \quad , \tag{14}$$

which is a first order differential equation with constant coefficients. And because it only has terms containing the unknown Q, it is also homogeneous.

Before we solve it, let's think like physicists and look at the units. The first term in the equation has units of charge per time, so the second term must have these units as well. The charge in the numerator is obvious, but time in the denominator? Yes; RC has units of time and this combination is called τ, the characteristic time of the circuit:

$$\tau = RC \quad . \tag{15}$$

Incidentally, since RC is a time then 1 Ohm $\times$ 1 Farad = 1 second. This means that checking the units in electricity and magnetism is a nightmare. The only way to keep it straight is to know simple formulas (like $\tau = RC$) and then to use them to figure out what the units of things are.

To solve the equation we make the usual guess that always works for linear homogeneous differential equations with constant coefficients:

$$Q(t) = Ae^{pt} \quad . \tag{16}$$

Substituting this guess into Eq. (14) and cancelling the common factor of e^{pt} then gives

$$p + \frac{1}{RC} = 0 \qquad \Rightarrow \qquad p = -\frac{1}{RC} = -\frac{1}{\tau} \quad . \tag{17}$$

This determines the unknown parameter p in the guess; to determine the other parameter A we use the initial conditions. If the initial charge on the capacitor is Q_o, then at time $t = 0$ we have

$$Q(0) = Q_o = Ae^0 = A \qquad \Rightarrow \qquad A = Q_o \quad . \tag{18}$$

So the solution to this circuit problem is

$$Q(t) = Q_o e^{-t/\tau} \quad \text{and} \quad I(t) = -\frac{dQ}{dt} = \frac{Q_o}{\tau} e^{-t/\tau} \quad . \tag{19}$$

A more physical way to write the current is to note that the voltage across the capacitor is given by $V(t) = Q(t)/C$ so we may write

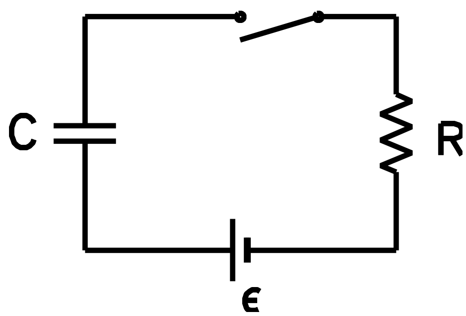
$$I(t) = \frac{Q(t)}{RC} = \frac{V(t)}{R} \tag{20}$$

our old friend, Ohm's law.

Hence, both the charge on the capacitor and the current through the resistor start out with their initial values, then exponentially decay in time with time constant $\tau = RC$. To see the physical meaning of the characteristic time, just wait for a time $t = \tau$ and see what the charge on the capacitor is:

$$Q(\tau) = Q_o e^{-1} = 0.37 Q_o \quad . \tag{21}$$

Hence, τ is the time required for the charge on the capacitor to decrease to 37% of its initial value.



2.2 Charging

To study the charging of a capacitor through a resistor, simply add a battery to the circuit, as shown above. Again we choose to traverse the circuit in the clockwise direction and we choose the direction of positive current to also be clockwise. We can't apply all of the rules yet because we don't know which plate of the capacitor is positive. It really doesn't matter which plate we designate as the positive one; we can just make a guess and if we guess wrong Q will turn out to be negative. But minus signs are confusing, so it's better if we choose well. To make a good choice, just imagine the switch closing and think about what will happen. The battery will start to pump clockwise current around the circuit, charging the lower plate of the capacitor positively. So we will let this lower plate be positive as we apply the loop rules. Traversing the circuit and using the rules then gives

$$\varepsilon - \frac{Q}{C} - IR = 0 \tag{22}$$

Again we write $I = \pm dQ/dt$, then look at the circuit to decide on the sign. This time positive current will put positive charge on the lower plate, causing Q there to increase, so $dQ/dt > 0$. Hence we choose $I = +dQ/dt$ in Eq. (22) to get

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{\varepsilon}{R} \quad . \tag{23}$$

The characteristic time $\tau = RC$ shows up again, so the solution of this equation will be similar to the last one. But this time the equation is inhomogeneous (the right-hand side isn't zero) so we have to remember how to handle such situations. The rule is that you write the solution as the sum of the particular solution and the homogeneous solution (the solution with zero for the right-hand side). The homogeneous equation is the same as for discharging, i.e.,

$$\frac{dQ}{dt} + \frac{Q}{RC} = 0 \quad , \tag{24}$$

so, its solution is the same:

$$Q_h(t) = Ae^{-t/\tau} \tag{25}$$

The particular solution is any solution of Eq. (23). Sometimes particular solutions are very hard to find, but a simple trick that often works is just to try a constant:

$$Q_p(t) = B \tag{26}$$

Substituting this guess for the particular solution into Eq. (23) gives

$$\frac{B}{RC} = \frac{\varepsilon}{R} \qquad \Rightarrow \qquad B = C\varepsilon \ \ . \tag{27}$$

Note that this particular solution has physical meaning in our case: it is the charge on a capacitor with voltage $V = \varepsilon$, which we expect to be the charge on the capacitor in our circuit after we have waited for a long time.

With Q_h and Q_p in hand we can put the total solution together:

$$Q(t) = Q_h(t) + Q_p(t) = Ae^{-t/\tau} + C\varepsilon \tag{28}$$

Applying the initial condition that the capacitor is uncharged when the switch is closed then gives

$$Q(0) = 0 = Ae^0 + C\varepsilon \qquad \Rightarrow \qquad A = -C\varepsilon \tag{29}$$

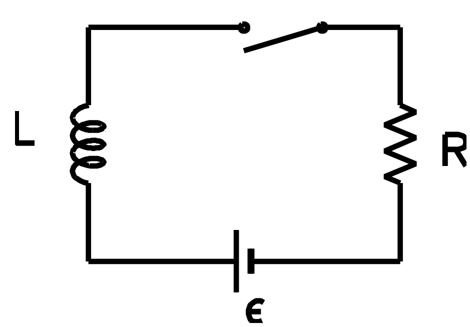
so we have finally

$$Q(t) = C\varepsilon(1 - e^{-t/\tau}) \quad ; \quad I(t) = \frac{dQ}{dt} = \frac{C\varepsilon}{\tau}e^{-t/\tau} = \frac{\varepsilon}{R}e^{-t/\tau} \tag{30}$$

Note that the charge rises from zero up to its final value, while the current starts at its large initial value of ε/R and decays away in time.

3 LR CIRCUITS

3.1 Switch On



Now let's look at a circuit with resistance and inductance in it. When we close the switch in the circuit shown above the battery starts to drive current through the resistor, with the inductor fighting the rising current a la Lenz's law. Using a clockwise loop, clockwise positive current, and the loop rules gives

$$\varepsilon - L\frac{dI}{dt} - IR = 0 \tag{31}$$

This is an inhomogeneous first order linear differential equation in the single variable $I(t)$:

$$\frac{dI}{dt} + \frac{I}{(L/R)} = \frac{\varepsilon}{L} \quad . \tag{32}$$

The characteristic time in this circuit is $\tau = L/R$, and since this equation is mathematically identical to Eq. (23) its solution is also identical:

$$I(t) = I_h(t) + I_p(t) \quad ; \quad I_h(t) = Ae^{-t/\tau} \quad ; \quad I_p(t) = \text{const} = \frac{\varepsilon}{R} \quad . \tag{33}$$

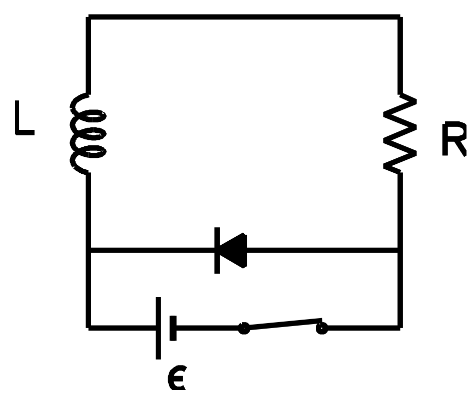
The particular solution I_p again has physical meaning: it is just the final current through the resistor after the current has settled down so that the inductor doesn't object anymore. To determine the final constant A we again use the initial conditions. This time it is that the initial current is zero, so

$$I(0) = 0 = Ae^0 + \frac{\varepsilon}{R} \qquad \Rightarrow \qquad A = -\frac{\varepsilon}{R} \tag{34}$$

so we have finally

$$I(t) = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) \quad . \tag{35}$$

3.2 Switch Off



Now we want to study what happens when you have steady current through an inductor, then turn it off. This is a little tricky to do. You have all had, or easily can have, the experience of trying suddenly to turn off the current through a big inductor: it's what happens when the vacuum cleaner is running and you jerk the plug out of the wall. When you do this you present the inductance of the electric motor with a dilemma. If it instantly kills the current then LdI/dt will be infinite, which is physically impossible. If it doesn't kill the current then LdI/dt is zero and since you have removed the driving emf, there is nothing to keep the current going. What's a circuit element to do? Well, it does one of two things. If the inductor is big enough it generates a large LdI/dt emf which makes a spark across the prongs of the plug, allowing the current to continue to flow while dropping to zero in a finite time (and making black burn marks on the outlet).

If the inductor is too small then the prongs of the plug act like a capacitor and the current oscillates, as discussed in section 4.1.

Let's concentrate here on big inductors. The spark generated by a big inductor when it is turned off is generally undesirable. For example, the motor might be running a conveyor belt in a coal plant or a flour mill where there is flammable dust in the air. Safety engineers are not impressed with electrical switches that power the equipment down by blowing the plant to bits. In the old days this problem was avoided by using mechanical trickery inside the switch, but in modern switches a simple diode is used, as shown in the circuit above. An ideal diode is just an electrical valve, acting like a perfect conductor for current flowing in the direction indicated by the diode arrow (forward bias) and acting like a resistor with $R = \infty$ for current flowing opposite to the arrow (reverse bias). Notice that with the switch closed the diode won't let any of the battery current flow through it, so it's just as if it weren't there. But when the switch is opened and the inductor wants to keep the current going, the diode is now biased forward and allows the current to run down without a spark. (If there is any justice in the world the person who invented this became very rich.)

Well, that's a long introduction to a really simple equation. After the switch has been opened we simply have an LR circuit with initial current $I_o = \varepsilon/R$. The loop rules give

$$-L\frac{dI}{dt} - IR = 0 \tag{36}$$

or

$$\frac{dI}{dt} + \frac{I}{(L/R)} = 0 \quad . \tag{37}$$

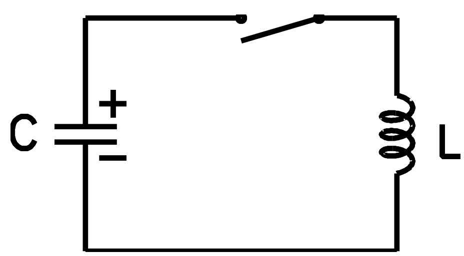
The characteristic time in this circuit is again $\tau = L/R$, and since this equation is mathematically identical to Eq. (14) its solution is also identical:

$$I(t) = I_o e^{-t/\tau} \quad . \tag{38}$$

So the current just runs down exponentially toward zero with time constant $\tau = L/R$.

4 LRC CIRCUITS

4.1 Undriven LC



The circuit shown above has a charged capacitor waiting to be discharged through an inductor. It is qualitatively different from the RC and LR circuits. In circuits containing resistance the energy in the circuit is dissipated as heat in the resistor, causing the current to approach some steady value late in time. In the LC circuit neither of the circuit elements can dissipate energy, so the circuit is *reactive*, or "bouncy", with energy being transferred back and forth between C and L. But in spite of this qualitative change of behavior, we still use the same kind of analysis to study the circuit. Take a clockwise loop and a clockwise positive current to get

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \quad ; \quad I = -\frac{dQ}{dt} \quad , \tag{39}$$

so

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \quad . \tag{40}$$

Because of the initial charge on the capacitor and because the inductor prevents current from flowing just after the switch is closed, the initial conditions for the differential equation are

$$Q(0) = Q_o \quad ; \quad \left.\frac{dQ}{dt}\right|_0 = -I(0) = 0 \tag{41}$$

4.1.1 Solving the Differential Equation

This time the differential equation is second order, but it's still linear and homogeneous, so we could use our standard guess of $Q(t) = Ae^{pt}$. But for reactive systems like this one it is a little simpler to anticipate that the solutions will oscillate by writing instead

$$Q(t) = Ae^{i\omega t} \quad . \tag{42}$$

To see why this represents an oscillation recall that

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \quad ; \quad \cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \quad ; \quad \sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \quad . \tag{43}$$

Hence, our usual oscillating sine and cosine functions are just linear combinations of $e^{\pm i\omega t}$.

Well, this probably sounds a little abstract, but it is actually very practical. To see how, let's apply this mathematics to the problem at hand. We substitute the guess $Q(t) = Ae^{i\omega t}$ (where A is in general complex) into Eq. (40) to get

$$A\frac{d^2e^{i\omega t}}{dt^2} + \frac{Ae^{i\omega t}}{LC} = 0 \quad \Rightarrow \quad A[(i\omega)^2 + \frac{1}{LC}]e^{i\omega t} = 0 \quad \Rightarrow \quad \omega = \pm\omega_0 \quad \text{where} \quad \omega_0 = \sqrt{\frac{1}{LC}} \tag{44}$$

From this point there are two ways to proceed. One is the standard mathematical procedure taught in courses on ordinary differential equations. The other is often used in time-dependent physics problems, but you may not have seen it before. I will show you both ways for the LC circuit, but will use the second method for the LCR circuit because it is easier and leads most naturally into AC circuits.

4.1.2 Standard Mathematical Approach

As usual for a second-order linear differential equation, there are two fundamental solutions, and the general solution is given as a linear combination of the two:

$$Q(t) = A_+e^{i\omega_0 t} + A_-e^{-i\omega_0 t} \tag{45}$$

The complex constants A_+ and A_- are determined by the initial conditions:

$$Q(0) = Q_o = A_+ + A_- \quad ; \quad \left.\frac{dQ}{dt}\right|_0 = 0 = i\omega_0 A_+ - i\omega_0 A_- \quad . \tag{46}$$

This gives two equations in two unknowns which are easily solved to obtain

$$A_- = A_+ \quad \text{so} \quad A_+ = A_- = \frac{Q_o}{2} \tag{47}$$

Hence we have for the solution

$$Q(t) = Q_o\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} = Q_o\cos\omega_0 t \quad , \tag{48}$$

where one of the identities in Eq. (43) has been used to obtain the final form.

4.1.3 Complex Amplitude Method

Another way to do this problem is to note that one of our fundamental solutions is actually a linear combination of two other fundamental solutions through DeMoivre's theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \Rightarrow \qquad e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t \tag{49}$$

(Note that the cosine and sine functions form a different set of two fundamental solutions.) The complex amplitude method takes advantage of this fact to make it feel like we are just working with one fundamental solution when we are really using two. To see how it works, consider the expression

$$Q(t) = Re\left[Ae^{i\omega_0 t}\right] \tag{50}$$

where A is a complex number. If we expand this expression using $A = A_r + iA_i$ we see that it really is a linear combination of the sine and cosine functions:

$$Q(t) = Re\left[Ae^{i\omega_0 t}\right] = A_r\cos\omega t - A_i\sin\omega t \tag{51}$$

The initial conditions determine the two parts of the complex constant A. But it is even better to represent A in polar form as

$$A = |A|e^{i\theta} \tag{52}$$

where $|A|$ is the amplitude and where θ is the phase.

Using this polar form the above expression for $Q(t)$ can be rewritten as

$$Q(t) = Re\left[Ae^{i\omega_0 t}\right] = |A|\cos(\omega_0 t + \theta) \tag{53}$$

As in the standard mathematical approach there are two constants to determine, but rather than the additive constants A_+ and A_-, the two are the amplitude and phase of the oscillation. For oscillating functions this is often a simpler way to approach the problem. For instance, in the LC circuit problem we are doing here the initial conditions and Eq. (53) require

$$Q(0) = Q_o = |A|\cos\theta \quad ; \quad \left.\frac{dQ}{dt}\right|_0 = 0 = |A|\sin\theta \tag{54}$$

This is easily solved by $\theta = 0$ and $|A| = Q_o$ giving

$$Q(t) = Q_o\cos\omega_0 t \ , \tag{55}$$

as obtained in 4.1.2.

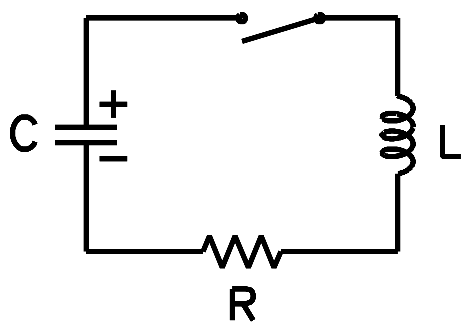
4.2 Undriven LRC

Now we will use this powerful method of complex amplitudes to solve a problem which is just awful using sines and cosines, but which is only moderately horrible using complex exponentials. The circuit is the LRC circuit shown above. Applying the loop theorem to this circuit after the switch is closed gives for the governing differential equation

$$\frac{Q}{C} - L\frac{dI}{dt} - IR = 0 \quad ; \quad I = -\frac{dQ}{dt} \ , \tag{56}$$

or

$$\frac{d^2Q}{dt^2} + \frac{1}{\tau}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \ , \tag{57}$$



where $\tau = L/R$. We again try a solution of the form

$$Q(t) = Ae^{i\omega t} \tag{58}$$

and substitute it into Eq. (57) to get

$$-\omega^2 + \frac{i}{\tau}\omega + \omega_0^2 = 0 \quad , \tag{59}$$

where $\omega_0 = 1/\sqrt{LC}$, as before. This quadratic equation has for its two solutions

$$\omega_+ = \frac{i}{2\tau} + \sqrt{\omega_0^2 - \frac{1}{4\tau^2}} \quad \text{and} \quad \omega_- = \frac{i}{2\tau} - \sqrt{\omega_0^2 - \frac{1}{4\tau^2}} \quad . \tag{60}$$

This problem has three different cases, depending on what happens with the square root in this expression. If the damping is weak ($1/2\tau \ll \omega_0$), then the square root is real, corresponding to the resistance-shifted oscillations frequency

$$\omega' = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}} \quad . \tag{61}$$

In this case we say that the system is underdamped, meaning that $Q(t)$ rings and gradually dies away. This is like a mass on a spring with weak air friction. If the damping is strong ($1/2\tau \gg \omega_0$) then the square root is imaginary and both of the fundamental solutions decay in time; the system is said to be overdamped This is like a mass on a spring in a bucket of tar. If the square root is exactly zero the system is said to be critically damped and you have to go look in a differential equation book to find what to do. Since this hardly ever happens we ignore the critically damped case. And since heavy damping is generally boring, we will assume in what follows that we have the underdamped case of a dying oscillation.

This problem can be done with the complex fundamental solutions $e^{i\omega_+ t}$ and $e^{i\omega_- t}$, but I will use the complex amplitude method instead. This method can be used whenever the two fundamental frequencies ω_+ and ω_- have the same imaginary parts and real parts which are opposite in sign, as is usually the case in problems where the solution is a damped (or undamped) oscillation. To use this method we choose one of the two frequencies, say ω_+, and write

$$Q(t) = Re\left[Ae^{i\omega_+ t}\right] = |A| Re\left[e^{i(\omega_+ t + \theta)}\right] \quad , \tag{62}$$

where we have used the polar form for the complex constant A, $A = |A|e^{i\theta}$, to get the second expression above. (We could also use ω_-——the final answer would turn out the same.)

With this form the initial conditions that $Q(0) = Q_0$ and $\dot{Q}(0) = 0$ become

$$Q_0 = |A|\cos\theta \quad \text{and} \quad 0 = |A| Re\left[i\omega_+ e^{i\theta}\right] \quad . \tag{63}$$

In the second of these conditions we can substitute the expression for ω_+ in Eq. (60), use $e^{i\theta} = \cos\theta + i\sin\theta$, and expand their product to obtain

$$\theta = -\tan^{-1}\left(\frac{1/(2\tau)}{\sqrt{\omega_0^2 - \frac{1}{4\tau^2}}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{4\omega_0^2\tau^2 - 1}}\right) \quad . \tag{64}$$

And now that we know θ the first condition just gives

$$|A| = \frac{Q_0}{\cos\theta} \tag{65}$$

Finally, we may use $\omega_+ = i/(2\tau) + \omega'$ in Eq. (62) to obtain

$$Q(t) = \frac{Q_0}{\cos\theta} e^{-t/(2\tau)} \cos(\omega' t + \theta) \quad . \tag{66}$$

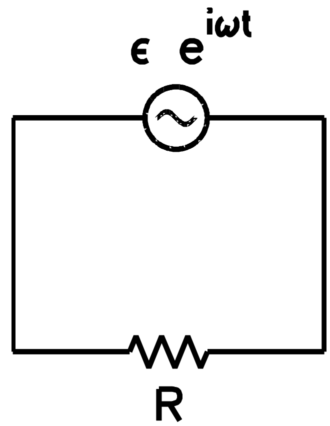
This may seem like a pretty terrible calculation, but if you think this is bad, you ought to see what it looks like using sines and cosines. Complex exponentials really are the best way to go, especially for the driven LRC circuits of the next, and final, section.

5 DRIVEN LRC CIRCUITS

We are now going to study driven AC circuits, and to do it right is quite a mess. So let's just talk qualitatively about what happens first, then focus our attention on the most important effects. When an AC driving emf of amplitude ε and frequency ω is connected to a circuit containing capacitance, inductance, and resistance things get a little crazy for a while. The driving emf is trying to push current around at its frequency ω and the ciruit is also trying to oscillate at its resistance-shifted frequency ω'. The result is beating between these two frequencies, giving rise to irregular, and sometimes wild, oscillations. This is part of the reason that the power company has a hard time turning the power back on after an outage. But if we think about how differential equations work, we can tell what will happen if we wait a while, because the general solution is a linear combination of the homogeneous solutions (the dying oscillations discussed in section 4.2) and the particular solution caused by the driving emf at frequency ω. The homogeneous solution is called a *transient*, because it dies away and becomes unimportant if we wait long enough. And what's left if we wait long enough? Just the particular solution driven by the AC emf. Rather than study this whole complicated business, we will restrict our attention to the steady-state particular solution. And to keep things as simple as possible we will first study an AC emf hooked up to single circuit elements. In the process we will discover that the behavior of single circuit elements is so simple (as long as you are comfortable with living in the complex plane) that complicated combinations of circuit elements can be handled with the same series-parallel rules you are used to using with DC circuits.

5.1 AC Resistor

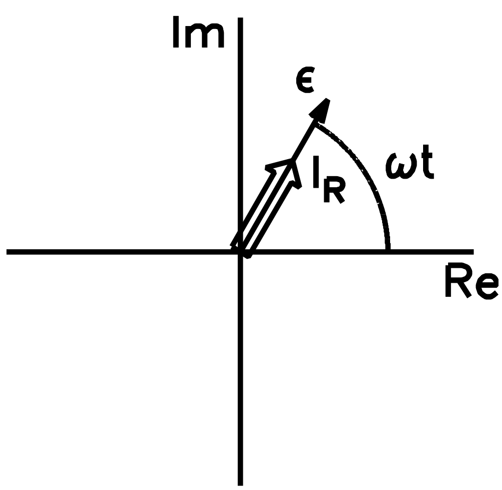
Consider the problem of an AC emf connected to a resistor, as shown above. Instead of representing the AC emf with the function $\cos\omega t$, we use our new friend $e^{i\omega t}$, by which we really mean its real part: $Re[e^{i\omega t}] = \cos\omega t$. Doing it this way means that we can take



advantage of the simple properties of the exponential function, then get the physical answer at the end by taking the real part. In this case we want to find the steady-state current I through the resistor driven by the emf. Applying our loop rules at a time when the current is clockwise gives

$$\varepsilon e^{i\omega t} - IR = 0 \qquad \Rightarrow \qquad I(t) = \frac{\varepsilon}{R} e^{i\omega t} \tag{67}$$

In Physics 122 you learned a nice pictorial way of seeing the relationship between driving emf and current: the phasor diagram. But if you had asked your instructor exactly what the plane was on which these phasor arrows were drawn you would have gotten some mumbling that didn't make any sense. You are now ready to understand what the phasor plane is. *It is the complex plane!* The complex phasor diagram for this resistor circuit is shown below.

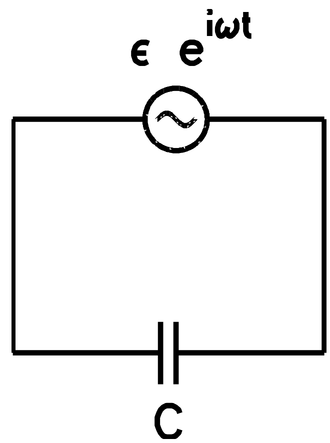


The angle between the real axis and the arrows is just the angle $\theta = \omega t$ that appears in $e^{i\theta} = e^{i\omega t}$, so when you look at a phasor diagram you are just looking at the complex voltages and currents in the complex plane. And all you have to do to convert these complex pictures into real physical data is to let the arrows rotate counterclockwise at frequency ω, then take the real parts of the complex arrows, which means to look at the projections of the phasor arrows down onto the real axis.

The complex plane is simply the best possible place to see the phase relationships between oscillating quantities, because phase shifts are actual angles between vectors, as we will see more clearly later. In the case of the resistor circuit it is easy to see that the driving emf and the current I exactly track each other in time, so there is no phase shift in this case. We summarize the response of this simple circuit to the driving emf by writing down the amplitude I_R of the current and the phase shift ϕ_R between the AC emf and the current:

$$I_R = \frac{\varepsilon}{R} \qquad ; \qquad \phi_R = 0 \qquad ; \qquad I(t) = I_R e^{i(\omega t + \phi_R)} \tag{68}$$

5.2 AC Capacitor



Now let's do something a little more interesting. Let's hook our AC emf up to a capacitor, as shown above. If we apply the loop theorem at an instant when the AC source is pumping current clockwise, charging the lower right capacitor plate with positive charge, then we get

$$\varepsilon e^{i\omega t} - \frac{Q}{C} = 0 \qquad \Rightarrow \qquad Q(t) = \varepsilon C e^{i\omega t} \tag{69}$$

This would be just fine if all we cared about was the charge on the capacitor, but our goal in developing AC circuit theory is to write everything in terms of the current in the circuit. Since positive (clockwise) current increases the charge on the lower right plate we have $I = dQ/dt$, so

$$I(t) = \frac{dQ}{dt} = i\varepsilon\omega C e^{i\omega t} \tag{70}$$

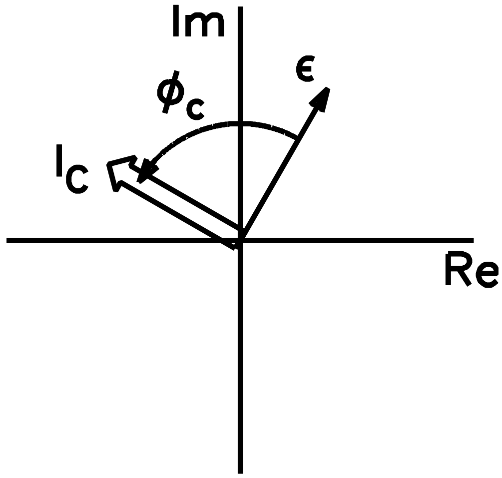
To see what the phasor diagram for this circuit looks like, simply recall that $i = e^{i\pi/2}$, so we may write

$$I(t) = e^{i\pi/2}\varepsilon\omega C e^{i\omega t} = \varepsilon\omega C e^{i(\omega t + \pi/2)} \tag{71}$$

Hence, the current arrow is out in front of the AC emf arrow by $\pi/2 = 90°$, as shown in the diagram below.

If it helps you remember this result, use the word ICE. The letter "C" in this word stands for capacitor, and in the word ICE "I" comes before "E", meaning that I is ahead in phase of the driving emf. Notice that the expression for the current in Eq. (71) looks very similar to that for the resistor case. The current has a certain amplitude, which is proportional to the driving emf, and there is a phase shift between the current and the driving emf. To take advantage of this similarity we define the *capacitive reactance* X_C to be a complex quantity with units of ohms that plays the same role in the capacitor circuit as that played by the resistance in the resistor circuit. To be precise, we define

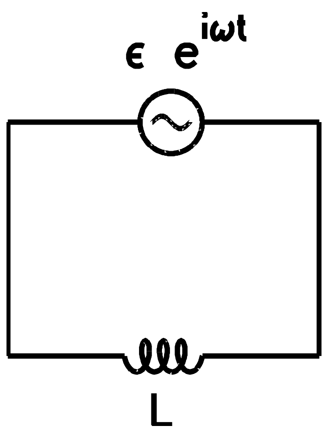
$$X_C = \frac{1}{i\omega C} \qquad \Rightarrow \qquad I(t) = \frac{\varepsilon}{X_C} e^{i\omega t} \tag{72}$$



so that the current amplitude I_C and the current phase shift ϕ_C can be written as follows:

$$I_C = \frac{\varepsilon}{|X_C|} \qquad ; \qquad \phi_C = \frac{\pi}{2} \qquad ; \qquad I(t) = I_C e^{i(\omega t + \phi_C)} \tag{73}$$

5.3 AC Inductor



OK, just one more circuit element to go. Let's hook our AC emf up to an inductor, as shown above. If we apply the loop theorem at an instant when the AC source is pumping current clockwise then we get

$$\varepsilon e^{i\omega t} - L\frac{dI}{dt} = 0 \qquad \Rightarrow \qquad \frac{dI}{dt} = \frac{\varepsilon}{L}e^{i\omega t} \tag{74}$$

Again we really want the current $I(t)$, so we have to solve this simple first order differential equation. The particular solution representing this steady state response is obtained by just integrating as usual;

$$I(t) = \int \frac{dI}{dt}dt = \frac{\varepsilon}{i\omega L}e^{i\omega t} + \text{const} \tag{75}$$

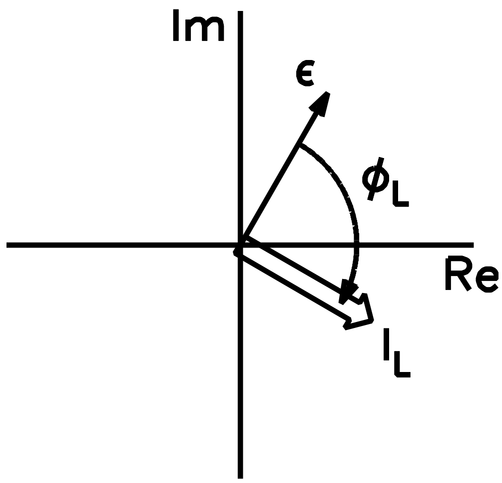
In this case the physical meaning of the constant of integration is a steady DC current, and since there is no source for such a current in our system, the constant must be zero.

$$I(t) = \frac{\varepsilon}{i\omega L} e^{i\omega t} \tag{76}$$

To see what the phasor diagram for this circuit looks like, simply recall that $i^{-1} = e^{-i\pi/2}$, so we may write

$$I(t) = \frac{\varepsilon}{\omega L} e^{i(\omega t - \pi/2)} \tag{77}$$

Hence, the current arrow is behind the AC emf arrow by $\pi/2 = 90°$, as shown in the diagram below.



To remember this result, use the word ELI. The letter "L" in this word stands for inductor, and in ELI "I" comes after "E", meaning that I is behind in phase compared to the driving emf. To remember both ICE and ELI, remember Eli the Iceman, who was put out of business by the AC electrical engineers who delivered cheap electricity to the new refrigerators in the homes of his former customers. (Technical people can be so cruel.)

As in the capacitor case, we define *inductive reactance* X_L to be a complex quantity with units of ohms that looks formally like resistance;

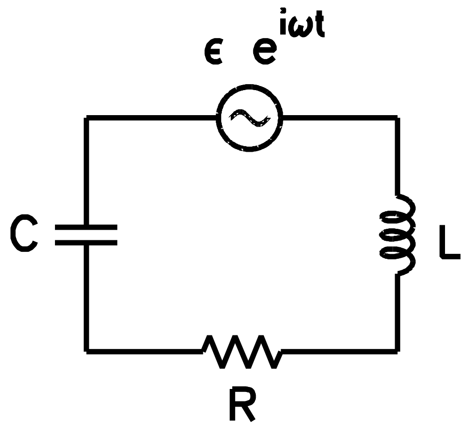
$$X_L = i\omega L \qquad \Rightarrow \qquad I(t) = \frac{\varepsilon}{X_L} e^{i\omega t} \tag{78}$$

Then the current amplitude I_L and the current phase shift ϕ_L can be written as follows:

$$I_L = \frac{\varepsilon}{|X_L|} \qquad ; \qquad \phi_L = -\frac{\pi}{2} \qquad ; \qquad I(t) = I_L e^{i(\omega t + \phi_L)} \tag{79}$$

5.4 Series LRC Circuit

The ideas of resistance, complex reactance, and phase shifts are the foundation for the study of more complex circuits. Consider, for example, the series LRC circuit shown above. As the AC emf drives current in this circuit things get a little crazy. Because the circuit elements are in series the current, both magnitude and phase, is the same in each element. But because of the reactive phase shifts of capacitors and inductors, the voltages across each element all have different phases. This means that when the voltage across the inductor is increasing



the voltage across the capacitor is decreasing, and the voltage across the resistor can be doing either. Somehow these phase-shifted voltages add together to match the negative of the driving emf so the loop theorem can be satisfied. In Physics 122 you handled this problem by carefully drawing diagrams in the phasor plane and using vector addition to make everything come out right.

But if we use the complex reactances discussed earlier in this section we can make this complicated analysis very simple. The reason is that vector addition can be done simply by adding the components of the vectors, and since complex numbers add in the same way as vectors (real parts add and imaginary parts add), *phasor vector arithmetic is simply the arithmetic of complex numbers.* What this means in practice is that AC circuits can be analyzed in exactly the same way as DC circuits, except that the additions involve complex numbers.

For example, to analyze the series circuit above we just add the resistances and reactances together just like we would if they were all resistors in series:

$$Z = R + X_C + X_L = R + \frac{1}{i\omega C} + i\omega L \quad . \tag{80}$$

The quantity Z is called the impedance of the circuit, and it is used just as we would use a total resistance, namely

$$I = \frac{\varepsilon}{Z} \quad . \tag{81}$$

But since Z is a complex number, I is now a complex number, so we had better stop for a bit to give meaning to the complex nature of I. The standard complex form for I would be

$$I = I_r + iI_i \quad . \tag{82}$$

This can be rewritten in polar form this way

$$I = |I|e^{i\theta} \quad \text{where} \quad |I| = \sqrt{I_r^2 + I_i^2} \quad ; \quad \theta = \tan^{-1}\left(\frac{I_i}{I_r}\right) \tag{83}$$

Now recall that the complete physical representation of the current, including the time dependence, is gotten from the real part of $Ie^{i\omega t}$:

$$I(t) = Re[Ie^{i\omega t}] = Re[|I|e^{i(\omega t+\theta)}] = |I|\cos(\omega t + \theta) \tag{84}$$

Hence, the magnitude of I is the amplitude of the oscillating current, and the polar angle θ is the phase shift of the current relative to the driving emf

$$\varepsilon(t) = \varepsilon \cos\omega t \quad . \tag{85}$$

In the case of the series circuit being considered here we have

$$|I| = \frac{\varepsilon}{|Z|} = \frac{\varepsilon}{\sqrt{R^2 + [1/(\omega C) - \omega L]^2}} \tag{86}$$

where we have used the fact that $X_C = \frac{-i}{\omega C}$, $X_L = i\omega L$, and $|Z| = \sqrt{Z_r^2 + Z_i^2}$. This is the standard series resonance formula for the current, showing that the current is maximum when the denominator is smallest, or when

$$\frac{1}{\omega C} = \omega L \qquad \Rightarrow \qquad \omega = \frac{1}{\sqrt{LC}} \quad , \tag{87}$$

i.e., when the driving frequency matches the natural frequency of the circuit ignoring the resistance. To get the formula for the phase shift we need to find the real and imaginary parts of I:

$$I = \frac{\varepsilon}{Z_r + iZ_i} = \frac{\varepsilon}{|Z|^2}(Z_r - iZ_i) \quad , \tag{88}$$

so we can write

$$\theta = \tan^{-1}\left(\frac{-Z_i}{Z_r}\right) = \tan^{-1}\left[\frac{1/(\omega C) - \omega L}{R}\right] \quad . \tag{89}$$

Notice that at resonance the phase shift is zero, i.e., another way to tell when this circuit is in resonance is that the current is in phase with the driving emf. And at this frequency the circuit behaves as if it were purely resistive.

Before moving on to a more interesting example I had better remind you of the "rms" business that shows up in oscillating systems. The amplitude of an oscillation is just the multiplier of $\cos(\omega t + \theta)$, but this number is not a particularly good indicator of how large the oscillation is because it is the largest possible value of the oscillating quantity. It is often more convenient to use some sort of an average amplitude to indicate the magnitude, but the usual average is no good because the oscillating quantity is negative as often as it is positive, so the average is zero. So the standard thing to do is to *Square* the oscillating quantity, take this new quantity's average, or *Mean*, then take the square *Root* to get back something with units of the original oscillation. Reading these keywords backwards gives *rms*, and for a simple sinusoidal oscillation the fact that the average value of $\cos^2 \omega t$ is $1/2$ means that the relation between the peak amplitude and the rms amplitude of an oscillation is

$$A_{rms} = \frac{A_{peak}}{\sqrt{2}} \quad . \tag{90}$$

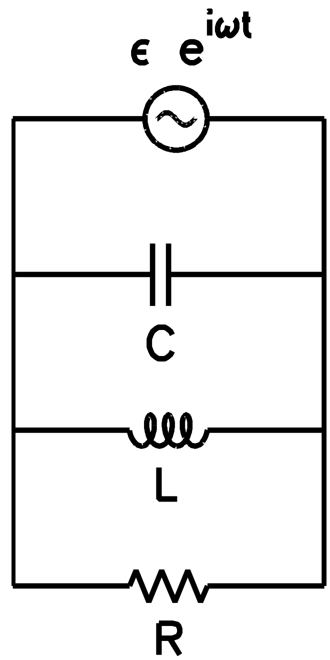
This rule doesn't work for products of oscillating quantities, however. For instance, since the power dissipated in a resistor is given by $I^2 R$, the relation between the peak power and the average power is

$$P_{avg} = R < I^2 >= R < I_{peak}^2 \cos^2 \omega t >= \frac{1}{2} I_{peak}^2 R = \frac{1}{2} P_{peak} \tag{91}$$

OK, that takes care of the simple series LRC circuit; now let's do something more interesting.

5.5 Parallel LRC Circuit

This parallel circuit can be analyzed with a phasor diagram similar to the series one. But this time it is the voltage that is the same across each element, and each voltage is the same



as that of the driving emf, both amplitude and phase. But the total current driven by the AC emf is the phasor sum of the currents that flow through each of the elements. This phasor analysis can be carried out using vectors, but it is easier just to treat each element as if it were a resistor and combine them into a total impedance just as we would in a DC circuit. Then we will find the total current I and analyze its complex nature to see what happens in this circuit. In so doing we will get exactly the same answer as we would have gotten by doing the more cumbersome phasor analysis.

The formula for combining three resistors in parallel is

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{92}$$

so we get the total impedance Z in the same way, using imaginary reactances for C and L:

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{i\omega L} + \frac{1}{1/(i\omega C)} = \frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right) \quad \Rightarrow \quad |Z| = \frac{1}{\sqrt{1/R^2 + [\omega C - 1/(\omega L)]^2}} \tag{93}$$

To find the total current we simply use

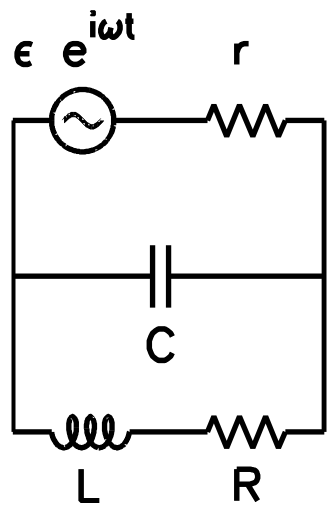
$$I = \varepsilon \frac{1}{Z} = \frac{\varepsilon}{R} + i\varepsilon\left(\omega C - \frac{1}{\omega L}\right) \quad , \tag{94}$$

showing clearly that the total current is simply the sum of the three currents flowing through each element. To find the magnitude and phase of the current we use Eq. (83):

$$|I| = \varepsilon\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \quad \text{and} \quad \theta = \tan^{-1}\left[\frac{\omega C - 1/(\omega L)}{1/R}\right] \quad . \tag{95}$$

Notice that resonance ($\omega = 1/\sqrt{LC}$) in this parallel circuit occurs in a different way than in the series circuit. In the series circuit resonance occurred when the driving emf drove a very large current (small impedance). In the parallel case resonance is signalled by the driving emf having a very small current (large impedance). The reason that small current means resonance in this case is that this small current is the current *supplied by the driving* *emf*. The capacitor and inductor still have large currents, but because the circuit is resonant, the driving emf is able to sustain large currents in L and C without doing hardly any work. The reason it can do this is that at resonance most of the current through L and C is just passed back and forth between them and the AC emf only has to supply the current that goes through the resistor. This points up the need for caution when looking for resonance: it may be indicated by either a maximum or a minimum in the impedance, depending on the circumstances.

5.6 Semi-complicated LRC Circuit



Now let's do something a little closer to the real world. Every AC emf has some internal resistance r (and inductance and capacitance as well–let's ignore these). Let's connect such a "real" AC generator to a real resonant circuit in which the real inductor has resistance R, as shown in the figure above. To find the impedance of this circuit we simply play the game of reducing each branch until a single impedance is obtained:

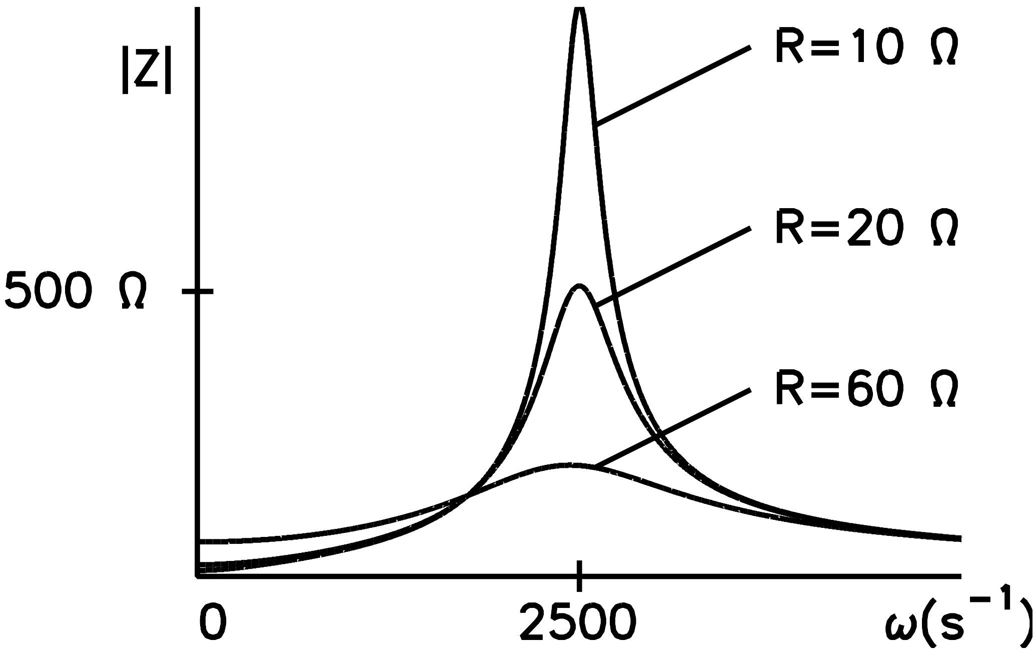
$$Z = r + \left(i\omega C + \frac{1}{R + i\omega L}\right)^{-1} \quad . \tag{96}$$

In complex form this expression for Z is quite simple, and if you have at your disposal a calculator or a math program (like Maple, Mathematica, or Matlab) that will handle complex arithmetic, it is even easy to use.

This example is actually quite interesting to study, but it is awkward with symbols for everything. So let's choose some values for the circuit elements.

$$L = 40 \text{ mH} \quad ; \quad C = 4\ \mu\text{F} \quad ; \quad r = .1\ \Omega \tag{97}$$

The figure below shows the $|Z(\omega)|$ for this circuit for several values of the inductor resistance R. Notice that resonance, which occurs at about $\omega = 1/\sqrt{LC} = 2.5 \times 10^3 \text{s}^{-1}$, occurs when $|Z|$ is maximum, as in the parallel circuit case. This is not surprising since the capacitor and the inductor are connected in parallel here as well. Notice that the resonant peak is narrower for small values of R, as also occurs in both the series and parallel circuit cases.

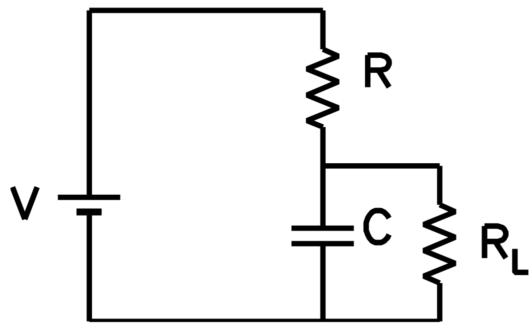


6 CONCLUSION

Well, that's enough reading; now it's time to work some problems on your own. Hopefully you now can see that the methods you learned to solve simple linear differential equations are just exactly what is needed to understand the behavior of electrical circuits. In particular, the functions e^{pt} and $e^{i\omega t}$ should now be your very special friends. And once you have mastered the complex arithmetic of reactance and impedance in AC circuits you should be able to analyze AC circuits almost as easily as you can analyze DC circuits.

7 PROBLEMS

1. The circuit shown below is a highway construction barricade flasher. The DC power supply charges the capacitor through the resistor R, but the cold neon light bulb R_L has such large resistance at first that no current can flow through it until the voltage across it reaches 9000 V. At this point the gas ionizes, R_L becomes quite small, the capacitor discharges through the lamp, and the lamp cools off again. After this rapid flash discharge, the power supply starts to charge the capacitor again. If $R = 1$ MΩ and if the DC power supply has $V = 12,000$ V, find the value of C that will make the flash occur every 3 seconds.



2. Think about an ordinary flashlight with a 3 V battery, a 5 W bulb, and the usual circuit wires. (Note that 5 W is the hot operating resistance of the filament, so when it is cold is resistance is a lot less.)

(a) Assume that the temperature of the hot tungsten filament when the bulb is using 5 W is 2500 K. Look up the resistance of tungsten vs. temperature and scale its hot resistance down to room temperature to obtain the resistance of the light bulb before the light is turned on.

(b) Estimate the inductance of the circuit. Problem 7.21 should be helpful in this part.

(c) Estimate the "time to turn on" of the flashlight. Could you ever notice this time delay when you turn on a flashlight?

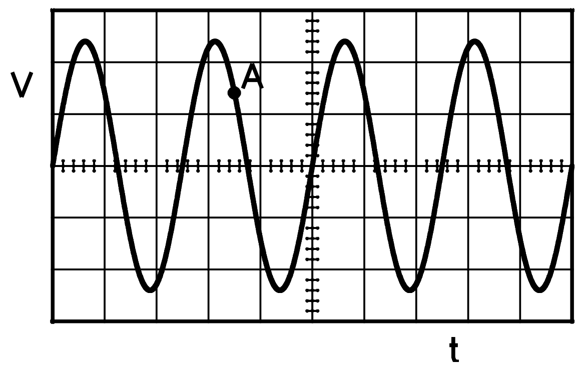
3. Convert the following complex numbers either to $x+iy$ form or to $re^{i\theta}$ form, depending on which form they are not already in. When converting to polar form be careful that your angles are in the range $-pi \leq \theta \leq \pi$. For each one, plot its position in the complex plane.

(a) $2 + 3i$ (b) $7e^{i\pi/3}$ (c) $-6 + 2i$ (d) $5e^{4i}$

4. An oscillation is described in complex form by the formula $Ae^{i\omega t}$ with $A = 3 - 5i$. By taking the real part of this complex form, find its physical form, i.e., convert it to the form $B\cos(\omega t + \phi)$ and find numerical values for B and ϕ. Hint: you had better use polar form.

5. The voltage across a 10 μF capacitor is shown on the oscilloscope trace below. The capacitor is connected with a solenoid of length 50 cm and radius 3 cm to form a series LC circuit. Ignore resistance and field-line fringing at the ends of the solenoid. Also ignore the thickness of the wires. Find the magnetic field at the center of the solenoid (inside) at the time marked A on the oscilloscope trace. Answer:

Vertical: 5 V/cm Horizontal: 200 μs/cm



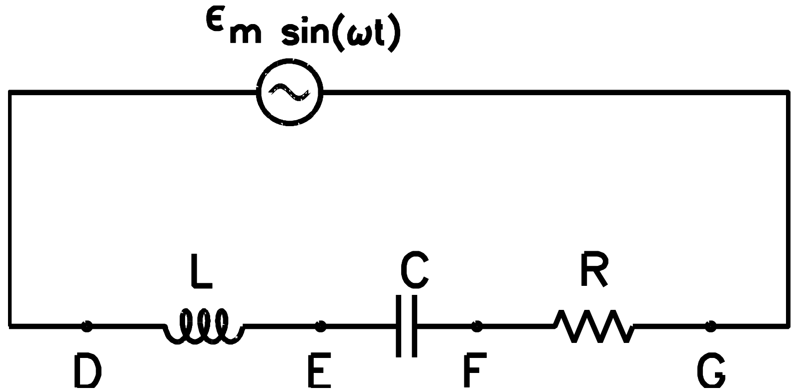
$B_A = 0.92$ mT

6. Derive the result for the phase angle θ in Sec. 4.2, Eq. (64), and also derive the final form for the solution in Eq. (66). Finally, differentiate Eq. (66) and show that at $t = 0$ it satisfies $\dot{Q}(0) = 0$.

7. Repeat the analysis of Sec. 4.1 but with the initial condition that at time $t = 0$ we have $Q = 0$ and $I = I_0$. Find $I(t)$.

8. Repeat the analysis of Sec. 4.2 but with the initial condition that at time $t = 0$ we have $Q(0) = Q_0$ and $I(0) = I_0$. Use the convention that Q is the charge on the upper plate of the capacitor and that I is positive when the current is clockwise. Find both $Q(t)$ and $I(t)$.

9. Here is our standard LRC circuit. Consider the oscilloscope traces that would be obtained by measuring various voltage differences. Between points F and G a signal $V = V_1 \cos(\omega t + \theta_1)$ would be obtained. Between points D and E a signal $V = V_2 \cos(\omega t + \theta_2)$ would be obtained. Between points E and G a signal $V = V_3 \cos(\omega t + \theta_3)$ would be obtained. Find expressions for V_1, V_2, V_3, θ_1, θ_2, and θ_3 involving the parameters given on the circuit diagram. You may also use correctly defined reactance and impedance symbols and the phase shift, θ, between the current and the driving emf, if you like.



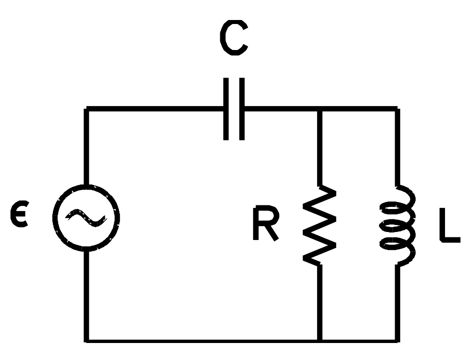
10. Consider the AC circuit shown below with

$\omega = 377\ \text{s}^{-1}$; $\varepsilon = 120$ V (rms) ; $R = 40\ \Omega$; $L = 53$ mH ; $C = 177\ \mu$F .

(a) Find the complex impedance Z and the phase angle θ. Tell whether I leads or lags the driving voltage.

(b) Make a plot of $|Z(\omega)|$ and see what kind of a resonance occurs, i.e., see if $|Z|$ is maximum or minimum at resonance.

(c) Make a plot of $\theta(\omega)$. Explain how θ behaves near resonance and see if the resonance points in $|Z|$ and θ are at the same frequency.



11. One of the demonstrations we do in Physics 122 is the circuit discussed in Sec. 5.6. We hook it up just the way it is described in that section, then sweep the frequency with a dial on the front of the generator. The oscilloscope leads are connected across the capacitor and we slowly change the driving frequency. We look for this voltage to go through a maximum at some frequency and then to come back down.

(a) Make a plot of the capacitor voltage as a function of frequency with $R = 20\ \Omega$ and $r = 0$, i.e., do it for a perfect AC generator with no internal resistance. Explain why the graph is so strikingly boring.

(b) Now increase the internal resistance r until you see the resonance appear in your plot. You will have to make it reasonably big to see the effect. Note that this means that the 122 classroom demonstration only works because we have lousy equipment for which r is big! Explain why a large internal resistance causes V_C to exhibit a resonance. (Hint: this is a parallel circuit, so resonance is when the current through the generator is a minimum.)