Critical Collapse of a Complex Scalar Field with Angular Momentum

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We report a new critical solution found at the threshold of axisymmetric gravitational collapse of a complex scalar field with angular momentum. To carry angular momentum the scalar field cannot be axisymmetric; however, its azimuthal dependence is defined so that the resulting stress-energy tensor and spacetime metric are axisymmetric. The critical solution found is nonspherical, discretely self-similar with an echoing exponent $\Delta = 0.42(\pm 4\%)$, and exhibits a scaling exponent $\gamma = 0.11(\pm 10\%)$ in near-critical collapse. Our simulations suggest that the solution is universal (within the imposed symmetry class), modulo a family-dependent constant, complex phase.

Introduction.—The main purpose of this work is to study the effect of angular momentum in axisymmetric critical collapse of massless scalar fields. Critical collapse refers to the threshold of black hole formation, where interesting effects known as critical phenomena [1] have been observed in the gravitational collapse of a wide variety of types of matter, as well as vacuum gravitational waves [2]. For spherically symmetric massless scalar collapse, this behavior includes universality, scale invariance, and power law scaling of length scales that arise near criticality. In supercritical collapse, the characteristic length is the mass, $M$, of black holes that form. In the case of rotating collapse, since angular momentum has dimension length$^2$, one might naively expect the angular momentum, $J$, of the black holes formed to scale as $J \approx M^2$. A more refined analysis carried out using perturbation theory [3] suggests that $J \approx M^{2(1 - \text{Re}[\lambda^2])}$, where $\text{Re}[\lambda]$ is the real part of the exponent $\lambda$ of the dominant perturbative mode that carries angular momentum. In [4], $\text{Re}[\lambda]$ was found to be roughly $-0.017$, implying an approximate scaling $J \approx M^{2.05}$. Thus, the Kerr parameter $a = J/M^2$ is expected to scale to zero (albeit slowly) as the black hole threshold is approached.

In general, numerical exploration of angular momentum in the collapse of a single real scalar field would require a 3D code, for axisymmetric distributions of such matter cannot carry angular momentum. Constructing a general relativistic 3D simulation capable of resolving the range of length scales that unfold in scalar field critical collapse is a daunting project, and may require computational capacity not currently available. A “cheaper” alternative is to consider a set of distinct scalar fields, each with azimuthal dependence and hence angular momentum, and then add the different fields coherently such that the net stress-energy tensor is axisymmetric. A natural way to achieve such a coherent sum is via a single complex field, as will be explained later (see [5] for an alternative approach). One drawback to this method is that imposing such an ansatz for the complex field forces a nonspherical energy distribution. This means that the class of solutions we can study occupies a region of phase space distinct from that of spherical spacetimes, and so we cannot explore the role of angular momentum as a perturbation in spherical critical collapse [1,6]. On a positive note, the fact that we do find a new (axisymmetric) critical solution is interesting aside from questions of angular momentum, because it suggests that phase space has a more intricate structure than one might have naively imagined, probably containing an infinite set of distinct intermediate attractors characterized by their behavior near the center of symmetry (the results of [5] are also in accord with this conjecture). Regarding the question of how net angular momentum affects threshold behavior in this model: It appears to be irrelevant, with the angular momentum of black holes formed in supercritical collapse decaying significantly faster than $J \approx M^2$. Below we briefly describe the physical system and code we use, and then present our results.

Physical system.—We consider the Einstein equations

$$R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi T_{\mu \nu},$$

(1)

where $g_{\mu \nu}$ is the spacetime metric, $R_{\mu \nu}$ is the Ricci tensor, and $R = R_{\mu \mu}$ is the Ricci scalar; we use geometric units with Newton’s constant $G$ and the speed of light $c$ set to 1. We also use a massless, minimally coupled, complex scalar field $\Psi$ (with complex conjugate $\bar{\Psi}$) as the matter source. $\Psi$ satisfies a wave equation $\Box \Psi + m^2 \Psi = 0$, and has a stress-energy tensor $T_{\mu \nu}$ given by

$$T_{\mu \nu} = \Psi_{,\mu} \bar{\Psi}_{,\nu} + \bar{\Psi}_{,\mu} \Psi_{,\nu} - g_{\mu \nu} \frac{1}{2} \bar{\Psi} \Psi,$$

(2)
We solve (1) and the wave equation (hereafter the field equations) in axisymmetry, using coordinates \([t, \rho, z, \phi]\), where \(\phi\) is adapted to the azimuthal symmetry, \(t\) is time-like, and \((\rho, z)\) reduce to standard cylindrical coordinates in the flat-space limit. The axial Killing vector is then
\[
\xi^\nu = \left( \frac{\partial}{\partial \phi} \right)^\nu.
\] (3)

The existence of this Killing vector allows us to define the conserved angular momentum, \(J\), of the spacetime
\[
J = - \int T_{\mu\nu} \xi^\mu n^\nu \sqrt{h} d^3x,
\] (4)
where the integration is over the \(t = \text{const}\) spacelike hypersurface, \(\Sigma\), \(h\) is the determinant of the intrinsic metric on \(\Sigma\), and \(n^\mu\) is the hypersurface normal vector. Using (2) and (3), (4) evaluates to
\[
J = - \int \left[ \Psi_\phi \bar{\Psi}_\phi + \bar{\Psi}_\phi \Psi_\phi \right] n^\nu \sqrt{h} d^3x.
\] (5)

Thus, for a configuration of the scalar field to have non-zero angular momentum, \(\Psi\) must have some azimuthal dependence. We thus adopt the following ansatz:
\[
\Psi(\rho, z, t, \phi) = \Phi(\rho, z, t) e^{im\phi},
\] (6)
where \(\Phi(\rho, z, t)\) is complex, and \(m\) must be an integer for the scalar field to be regular. It is straightforward to check that this form of \(\Psi\) gives a stress-energy tensor that is conserved angular momentum, \(J\), and that this form of \(\Phi\) gives a stress-energy tensor that is zero angular momentum, \(J\), and that this form of \(\Phi\) gives a stress-energy tensor that is the hypersurface normal vector. We separately evolve the real and imaginary components of the scalar field by defining real functions \(\Phi_r\) and \(\Phi_i\) via
\[
\Phi = \rho(\Phi_r + i\Phi_i),
\] (9)
and their dynamical conjugates \(\Pi_r\) and \(\Pi_i\) by
\[
\Pi_r = \text{Re}[\Phi_\alpha \Phi_\alpha^*] / \rho, \quad \Pi_i = \text{Im}[\Phi_\alpha \Phi_\alpha^*] / \rho.
\] (10)

The factors of \(\rho\) appearing in the above definitions are included so that \(\Phi_r, \Phi_i, \Pi_r\), and \(\Pi_i\) satisfy Neumann conditions on axis. Similarly, the variables corresponding to \(\omega_\alpha\) and \(\xi_\alpha\) that are evolved in the code have appropriate powers of \(\rho\) factored out so that they satisfy Dirichlet conditions on axis (see [10] for the specific definitions).

We use the following initial data for the scalar field:
\[
\Phi_{\alpha i}(\rho, z, 0) = A_{\alpha i} \exp[-(\sqrt{\rho^2 + z^2} - R_{\alpha i})^2 / \Sigma_{\alpha i}^2],
\]
\[
\Pi_{\alpha i}(\rho, z, 0) = e_{\alpha i} \Phi_{\alpha i}(\rho, z, 0),
\] (11)
where \(A_{\alpha}, A_i, R_{\alpha}, R_i, \Sigma, \epsilon_r, \epsilon_i\), and \(e_i\) are parameters fixing the shape of the initial scalar field profiles. All other freely specifiable variables are set to zero at \(t = 0\), while the constrained variables \(\alpha, \psi\), and \(\{\beta^\rho, \beta^z, \nu_\rho\}\) are obtained by solving the maximal slicing condition, Hamiltonian, and momentum constraints, respectively.

We use a partially constrained finite-difference scheme with adaptive mesh refinement (AMR) to evolve the system of equations with time. In particular, the slicing condition and momentum constraints are used to fix \(\alpha\) and \(\{\beta^\rho, \beta^z\}\), respectively, while the remainder of the variables are updated using their evolution equations.

**Results.**—We now present results from a preliminary study of the black hole threshold of the complex scalar field system introduced in the previous section. We focus on four sets of initial data, summarized in Table I. Family \(A\) consists of identical pulses of \(\Phi_r(\rho, z, 0)\) and \(\Phi_i(\rho, z, 0)\) that are initially approximately outgoing \((\epsilon_i = 1)\) and ingoing \((\epsilon_i = -1)\), respectively. This choice for \((\epsilon_r, \epsilon_i)\) in a sense maximizes the net angular momentum \((5)\), given the initial profiles for \(\Phi_r\) and \(\Phi_i\). Conversely, family \(B\) is time symmetric, and hence has zero net angular momentum. Families \(A\) and \(B\) can be written as

<table>
<thead>
<tr>
<th>Label</th>
<th>(p)</th>
<th>(R_r)</th>
<th>(R_y)</th>
<th>(\Sigma)</th>
<th>(\epsilon_r)</th>
<th>(\epsilon_i)</th>
<th>(\delta_0)</th>
<th>(\delta^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(A_r = A_i)</td>
<td>0.6</td>
<td>0.6</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(\pi/4)</td>
</tr>
<tr>
<td>(B)</td>
<td>(A_r = 3A_i)</td>
<td>0.6</td>
<td>0.6</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\pi/4)</td>
</tr>
<tr>
<td>(C)</td>
<td>(A_r = A_i)</td>
<td>0.65</td>
<td>0.6</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(\tan^{-1} 3)</td>
</tr>
</tbody>
</table>
\[ \Phi(\rho, z, 0) = A(\rho, z)e^{i\delta_0}, \]

where \( A(\rho, z) \) is a real amplitude function and \( \delta_0 \) is a constant phase factor, equal to \( \pi/4 \) in both cases. Family C is thus identical to family A except for the initial phase. For family D, \( \Phi_r \) and \( \Phi_i \) have distinct initial profiles and thus cannot be characterized by a constant phase.

Based upon the collapse simulations we have performed for these four families of initial data, we can suggest the following about the threshold behavior for this matter model. There is apparently a discretely self-similar critical solution that is universal to within a family-dependent phase. In other words, one can write the critical solution \( \Phi^* \) for the scalar field as \( \Phi^*(\rho, z, t) = A^*(\rho, z, t)e^{i\delta^*} \), where \( A^*(\rho, z, t) \) is a universal real function and \( \delta^* \) is a family-dependent constant (see Table I). Note that this phase dependence is a consequence of the U(1) symmetry of the Lagrangian of the complex field, and has been observed in charged scalar field critical collapse [11]. Also, note that any self-similar solution is unique only up to a global rescaling of the form \( (\tilde{t}, \tilde{\mathbf{x}}) \rightarrow (\kappa \tilde{t}, \kappa \tilde{\mathbf{x}}) \) when written in suitable coordinates \( (\tilde{t}, \tilde{\mathbf{x}}) \), with \( \kappa \) a constant. Figure 1 shows a snapshot of the real part of the scalar field \( (\Phi_r, \rho) \) at late times in a near-critical collapse simulation. To estimate \( \delta^* \) for a given family, and the echoing exponent \( \Delta \) for the putative critical solution, we examine the central value of the real and imaginary parts of the scalar field divided by proper radius \( \rho_c \equiv \rho \psi^2 e^{i\sigma} \) (to factor out the leading order approach to zero of \( \Phi \) in a covariant manner). Figure 2 shows plots of \( \Phi_r \rho/\rho_c \) and \( \Phi_i \rho/\rho_c \) versus \( -1n\tau \) for the nearest-to-threshold solutions found, where \( \tau \) is central proper time (calculated as in [7]), defined such that the accumulation event of the critical solution corresponds to \( \tau = 0 \). We have multiplied the scalar field by \( \tau \) to cancel the artificial growth introduced by dividing by \( \rho_c \). Note that the equations of motion for \( \Phi_r \) and \( \Phi_i \) are identical; hence if \( \Phi_i(\rho, z, 0) = \Phi_i(\rho, z, 0) \) (as with family B), then the initial phase, \( \delta_0 = \pi/4 \), is preserved during evolution. The echoing exponent \( \Delta \) is the period of the self-similar solution in logarithmic proper time; from Fig. 2 (and similar data for family B) we estimate \( \Delta = (0.42 \pm 4\%) \). To estimate the scaling exponent \( \gamma \), we measure how the maximum value attained by the Ricci scalar (on axis), \( R_m \), in subcritical evolutions depends upon the parameter-space distance from threshold, \( p^* - p \) [12]. Representative results are shown in Fig. 3. Combining such data from all the families, we estimate \( \gamma = (0.11 \pm 10\%) \). For a discretely self-similar solution, one expects the linear relationship assumed in Fig. 3 to be modulated by an oscillation of period \( 2\Delta \) [13]; we have not run a sufficient number of simulations to adequately resolve such an oscillation. The uncertainties quoted
holes formed in supercritical collapse. $M$ time), Fig. 4 suggests that
of points to the left and right of the ‘‘knee” in the curve at
ln/0.0022, with slopes $\approx 6.0$ and $\approx 2.2$, respectively. In
ln$\rho$, the horizontal scale ranges from $-22$ to $-2$ (compare to
Fig. 3).

above for $\gamma$, $\Delta$, and $\delta^*$ (in Table I) were estimated from
convergence calculations from simulations using three
different values of the maximum truncation error thresh-
hold that controls the AMR algorithm, but do not account
for possible systematic errors (see a discussion of related
issues in [7]).

Regarding the question of how angular momentum
affects critical collapse: For the initial data described
here, net angular momentum seems to be completely
irrelevant. To within the accuracy of our simulations, we
cannot differentiate between the late time, self-similar
regions of the spacetimes obtained from families A and
B, and, in the latter case, there is no angular momentum.
Figure 4 shows a plot of the mass estimate $M_{AH}$ versus
angular momentum $J_{AH}$ on a logarithmic scale, of black
holes formed in supercritical collapse. $M_{AH}$ and $J_{AH}$ are
calculated from the area and angular momentum of the
apparent horizon, respectively (using the dynamical ho-
Figure 4 suggests that $J_{AH} \approx M_{AH}^2$. However, this
region of parameter space is “maximally” far from
threshold, in that these are almost the largest black holes
that we can form from initial data not already containing
an apparent horizon. For somewhat smaller black holes,
Fig. 4 shows a transition to a relationship closer to $J_{AH} \approx
M_{AH}^2$. However, we are still far from threshold there,
and furthermore are entering the regime where the angular
momentum calculation is dominated by numerical errors;
hence we cannot be certain about the exact value of the
exponent.

In summary, within the context of the class of complex
scalar field configurations subject to an azimuthal depen-
dence given by (6), we have found that in near-threshold

gravitational collapse net angular momentum scales to
zero significantly faster than the $J \propto M^2$ one expects from
dimensional analysis. However, we do observe critical
behavior at threshold, and this is only the second, non-
spherical critical solution found to date, the other being
that seen in the collapse of gravitational waves [15]. What
is remarkable about the new critical solution is that it
exists within a model permitting spherical critical solu-
tions (which is not the case for gravitational waves). This
suggests that the threshold of scalar field black hole
formation is a much more interesting regime than pre-
viously thought, possibly containing infinitely many dis-
tinct solutions.

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