In defense of the Morfey-Howell single-point nonlinearity indicator: An impedance-based interpretation

Won-Suk Ohm, Kent L. Gee, and Brent O. Reichman

Citation: Proc. Mtgs. Acoust. **29**, 045003 (2016); doi: 10.1121/2.0000427
View online: http://dx.doi.org/10.1121/2.0000427
View Table of Contents: http://asa.scitation.org/toc/pma/29/1
Published by the Acoustical Society of America

Proceedings of Meetings on Acoustics

Volume 29 http://acousticalsociety.org/

172nd Meeting of the Acoustical Society of America
Honolulu, Hawaii
28 November - 02 December 2016

Physical Acoustics: Paper 2pPA9

In defense of the Morfey-Howell single-point nonlinearity indicator: An impedance-based interpretation

Won-Suk Ohm
School of Mechanical Engineering, Yonsei University, Seoul, Korea; ohm@yonsei.ac.kr

Kent L. Gee and **Brent O. Reichman**
Department of Physics and Astronomy, Brigham Young University, Provo, UT; kentgee@byu.edu, brent.reichman@gmail.com

Since the Morfey-Howell *Q*/*S* was proposed as a nonlinearity indicator for propagation of intense broadband noise [AIAA J. **19**, 986-992 (1981)], there has been considerable debate as to its meaning and utility. Perhaps the most contentious argument against *Q*/*S* is about its validity as a single-point nonlinearity indicator: the importance of nonlinearity is often judged by observing cumulative effects over some propagation distance, whereas *Q*/*S* is based on a pressure waveform at a single location. Studies to address these criticisms have emerged over the years, most recently by Reichman et al. [J. Acoust. Soc. Am. **139**, 2505-2513 (2016)] in support of *Q*/*S*. In this paper, we show that the Burgers equation (from which *Q*/*S* was originally derived) can be recast in terms of specific impedance, linear absorption and dispersion coefficients, and normalized quadspectral (*Q*/*S*) and cospectral (*C*/*S*) densities. The resulting interpretation is that *Q*/*S* and *C*/*S* represent the additional absorption and dispersion, introduced by the passage of a finite-amplitude wave to the existing linear absorption and dispersion. In other words, a nonlinear wave process alters the apparent material properties of the medium, the extent of which can be used as a single-point indicator of the relative strength of nonlinearity.

Published by the Acoustical Society of America

1. INTRODUCTION

A nonlinearity indicator is a measure by which the strength of nonlinearity in a wave process is quantified. Examples of nonlinearity indicators include the Gol'dberg number,[1] the average steepening factor,[2] the derivative skewness,[3] the bispectrum,[4] and the Morfey-Howell *Q*/*S*.[5] Among these, the Morfey-Howell *Q*/*S*, originally proposed as a statistical measure of nonlinearity in aircraft jet noise, seems to invite more criticisms than any other indicators as to its validity, utility, and interpretation. The arguments leveled against *Q*/*S* can be summarized as follows: (a) the Burgers equation, from which *Q*/*S* is derived, is not applicable to fully 3-D jet noise (validity), (b) because *Q*/*S* cannot be marched numerically as a function of propagation distance, it is of little practical value (utility), and (c) the definition of *Q*/*S* is so arcane that it is difficult to grasp its physical meaning (interpretation).

The first two criticisms seem unwarranted. First, the same *Q*/*S* can be derived from the Westervelt equation that fully accounts for the 3-D wave structure (the subject of a forthcoming paper). In fact, *Q*/*S* is an absolute indicator of nonlinearity, which is independent of such linear wave processes as absorption/dispersion, geometrical spreading, refraction, and diffraction. Second, *Q*/*S* is an *indicator* of nonlinearity just like the rest of the aforementioned nonlinearity indicators, and thus it would be unfair to single out *Q*/*S* for its lack of predictive utility. For example, no criticism has ever been directed at the Gol'dberg number in this regard.

Apart from the machinery of applying *Q*/*S*, however, a fundamental understanding of its physical meaning has remained more or less elusive: what does *Q*/*S* really mean? Recently, Reichman et al.[6] offered a fresh, intuitive look at *Q*/*S* by repackaging the Burgers equation, in which *Q*/*S* is interpreted as the "additional change in level" of a spectral component due to nonlinearity. In addition to the thermoviscous loss, a spectral component can experience extra loss (or gain) in level through the nonlinear energy exchange with other spectral components, the extent of which is quantified by *Q*/*S*. In this paper, we aim to provide an alternative, but equally intuitive interpretation *Q*/*S* based on the familiar concept of specific acoustic impedance. These interpretations go to show that the physical meaning of *Q*/*S* is rather straightforward, and hence they would serve to promote the wider use of *Q*/*S* in nonlinear acoustics.

2. REPACKAGING THE BURGERS EQUATION

In order to arrive at the impedance-based interpretation of *Q*/*S*, one must start with the time-domain Burgers equation[7]

$$\frac{\partial p}{\partial x} = l_\tau(p) + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau}, \tag{1}$$

where p is the acoustic pressure, x is the propagation distance, τ is the retarded time, ρ_0 is the density, c_0 is the small-signal sound speed, β is the coefficient of nonlinearity, and $l_\tau(p)$ is the general absorption/dispersion operator. The corresponding spectral version of the Burgers equation is

$$\frac{\partial \tilde{p}}{\partial x} = l_\omega(\tilde{p}) + \frac{j\omega\beta}{2\rho_0 c_0^3}\tilde{q}\,, \tag{2}$$

where $\tilde{p}$ and $\tilde{q}$ are the Fourier transforms of the acoustic pressure and the squared acoustic pressure, respectively, and ω is the angular frequency. Here, the Fourier equivalent of the absorption/dispersion operator can be written as

$$l_\omega(\tilde{p}) = -(\alpha + j\delta)\tilde{p}\,, \tag{3}$$

where α and δ are frequency-dependent absorption and dispersion coefficients, respectively. The spectral Burgers equation then becomes

$$\frac{\partial \tilde{p}}{\partial x} = -(\alpha + j\delta)\tilde{p} + \frac{jk\beta}{2\rho_0 c_0^2}\tilde{q}\,. \tag{4}$$

Rewriting Eq. (4) in terms of the specific acoustic impedance

$$Z = \frac{\tilde{p}}{\tilde{u}} \tag{5}$$

requires the spectral version of the linearized Euler equation that connects the pressure gradient $\partial\tilde{p}/\partial x$ and the particle velocity $\tilde{u}$. We begin with the linearized Euler equation in *nonretarded* time *t*:

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0\,, \tag{6}$$

where u is the particle velocity. To recast Eq. (6) in retarded time τ consider the coordinate transformation[7]

$$x_1 = \tilde{\varepsilon}x, \qquad \tau = t - x/c_0\,. \tag{7}$$

Here, x_1 is the slow scale corresponding to the retarded time frame τ, and $\tilde{\varepsilon}$ is a small ordering parameter. Partial derivatives in the transformed coordinates (x_1, τ) are then

$$\frac{\partial}{\partial x} = \tilde{\varepsilon}\frac{\partial}{\partial x_1} - \frac{1}{c_0}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\,. \tag{8}$$

Substitution of Eqs. (8) into Eq. (6) yields

$$\tilde{\varepsilon}\frac{\partial p}{\partial x_1} = \frac{1}{c_0}\frac{\partial p}{\partial \tau} - \rho_0 \frac{\partial u}{\partial \tau}. \tag{9}$$

By replacing $\tilde{\varepsilon}(\partial p/\partial x_1)$ in Eq. (9) with $\partial p/\partial x$, the linearized Euler equation in retarded time τ is obtained:

$$\frac{\partial p}{\partial x} = \frac{1}{c_0}\frac{\partial p}{\partial \tau} - \rho_0 \frac{\partial u}{\partial \tau}. \tag{10}$$

Note that $\partial p/\partial x$ in Eq. (10) is of $O(\tilde{\varepsilon}^2)$, whereas $\partial p/\partial x \sim O(\tilde{\varepsilon})$ in Eq. (6). The Fourier transform of Eq. (10) gives the spectral version of the linearized Euler equation

$$\frac{\partial \tilde{p}}{\partial x} = jk\tilde{p} - j\omega\rho_0\tilde{u}, \tag{11}$$

where k is the wave number. Manipulation of Eq. (11) leads to an expression containing the dimensionless impedance $\bar{Z} = Z/\rho_0 c_0$:

$$\begin{aligned}\frac{\partial \tilde{p}}{\partial x} &= jk\tilde{p}\left(1 - \frac{j\omega\rho_0\tilde{u}}{jk\tilde{p}}\right) = jk\tilde{p}\left(1 - \frac{\rho_0 c_0}{\tilde{p}/\tilde{u}}\right) \\ &= jk\tilde{p}\left(1 - \frac{1}{\bar{Z}}\right).\end{aligned} \tag{12}$$

Furthermore, the term within the parenthesis in Eq. (12) can be expressed as, via binomial expansion in $\Delta\bar{Z} = \bar{Z} - 1$,

$$\begin{aligned}1 - \frac{1}{\bar{Z}} &= 1 - (1 + \Delta\bar{Z})^{-1} = 1 - \{1 - \Delta\bar{Z} + O(\tilde{\varepsilon}^2)\} \\ &= \Delta\bar{Z} + O(\tilde{\varepsilon}^2).\end{aligned} \tag{13}$$

Therefore, the spectral version of the linearized Euler equation at $O(\tilde{\varepsilon}^2)$ is given by

$$\frac{\partial \tilde{p}}{\partial x} = jk\tilde{p}(\bar{Z} - 1). \tag{14}$$

Now substitute Eq. (14) into the spectral Burgers equation [Eq. (4)] to obtain

$$jk\tilde{p}(\bar{Z}-1)=-(\alpha+j\delta)\tilde{p}+\frac{jk\beta}{2\rho_0 c_0^2}\tilde{q}\,. \tag{15}$$

Multiplication of Eq. (15) by $\tilde{p}^*$ gives

$$|\tilde{p}|^2(\bar{Z}-1)=j\frac{\alpha}{k}|\tilde{p}|^2-\frac{\delta}{k}|\tilde{p}|^2+\frac{\beta}{2\rho_0 c_0^2}\tilde{p}^*\tilde{q}\,. \tag{16}$$

Ensemble-averaging Eq. (16) leads to

$$S_{pp}(\bar{Z}-1)=j\frac{\alpha}{k}S_{pp}-\frac{\delta}{k}S_{pp}+\frac{\beta}{2\rho_0 c_0^2}\left(C_{pp^2}+jQ_{pp^2}\right), \tag{17}$$

where $S_{pp}=E\left[|\tilde{p}|^2\right]$, $C_{pp^2}=\mathrm{Re}\{E[\tilde{p}^*\tilde{q}]\}$, and $Q_{pp^2}=\mathrm{Im}\{E[\tilde{p}^*\tilde{q}]\}$ are referred to as the autospectrum, the cospectrum, and the quadspectrum, respectively. The final form of the ensemble-averaged, spectral Burgers equation reads

$$(\bar{Z}-1)=j\frac{\alpha}{k}-\frac{\delta}{k}+\frac{\beta}{2\rho_0 c_0^2}\left(\frac{C_{pp^2}}{S_{pp}}+j\frac{Q_{pp^2}}{S_{pp}}\right), \tag{18}$$

or, in real and imaginary parts,

$$\mathrm{Re}[\bar{Z}]-1=-\frac{\delta}{k}+\frac{\beta}{2}\frac{C}{S} \tag{19}$$

and

$$\mathrm{Im}[\bar{Z}]=\frac{\alpha}{k}+\frac{\beta}{2}\frac{Q}{S}\,. \tag{20}$$

Here, we introduce two nonlinearity indicators C/S and Q/S defined by

$$\frac{C}{S}=\frac{C_{pp^2}}{\rho_0 c_0^2 S_{pp}}, \qquad \frac{Q}{S}=\frac{Q_{pp^2}}{\rho_0 c_0^2 S_{pp}}\,. \tag{21}$$

Note that the Q/S in Eqs. (21) is equivalent to the Morfey-Howell Q/S up to a normalization constant: in Eq. (21) the bulk modulus of the medium $\rho_0 c_0^2$ is the normalization constant, whereas in the definition of Morfey-Howell the root-mean-square pressure p_{rms} is used.[5,6]

3. INTERPRETATION OF *Q*/*S* AND *C*/*S*

For plane progressive waves in linear acoustics, the specific acoustic impedance [Eq. (5)] can be construed as a medium property that a wave "sees" at any given point. For instance, a wave propagating in an ideal, lossless medium would see the specific impedance given by the characteristic impedance $\rho_0 c_0$ (or unity in dimensionless impedance):

$$\mathrm{Re}[\bar{Z}]-1=0, \qquad \mathrm{Im}[\bar{Z}]=0. \tag{22}$$

Any deviation of the dimensionless impedance from unity then signifies the presence of loss mechanisms such as absorption and dispersion. If the medium is lossy, a progressive wave sees the dimensionless impedance different from unity by the amount commensurate with the strength of absorption and dispersion. This is immediately apparent from Eqs. (19) and (20) without the *C*/*S* and *Q*/*S* terms:

$$\mathrm{Re}[\bar{Z}]-1=-\frac{\delta}{k}, \qquad \mathrm{Im}[\bar{Z}]=\frac{\alpha}{k}. \tag{23}$$

What if the finite-amplitude effects are taken into account? The significance of Eqs. (19) and (20) is that they provide a framework within which the quantities *C*/*S* and *Q*/*S* can be interpreted as the "additional change in impedance" due to nonlinearity. It follows from Eqs. (19) and (20) that *Q*/*S* and *C*/*S* represent the parametrically-induced change in impedance in the form of extra absorption and dispersion. Here, the passage of a finite-amplitude wave alters the apparent medium property, the extent of which can be used to quantify the strength of nonlinearity.

Finally, a companion nonlinearity indicator *C*/*S* is introduced for the first time. Examination of Eqs. (19) and (20) indicates that *C*/*S* and *Q*/*S* are complementary (i.e., *C*/*S* is to dispersion as *Q*/*S* is to absorption), and together, they constitute a complete set of nonlinearity indicators for finite-amplitude waves in media with general absorption and dispersion laws. Note that only the thermoviscous absorption is considered in the original derivation of the Morfey-Howell *Q*/*S*.

For dispersion-dominant systems, it is recommended that *C*/*S* should be used in place of *Q*/*S* as a nonlinearity indicator. For example, consider a wave system governed by the Korteweg-de Vries (KdV) equation:[8]

$$\frac{\partial p}{\partial x}=d\,\frac{\partial^3 p}{\partial \tau^3}+\frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau}, \tag{24}$$

where d is the dispersion parameter. In Eq. (24) absorption is assumed to be zero, and dispersion exhibits a cubic dependence on frequency. Wave systems with KdV-type dispersion include incompressible waves on the liquid surface[8] and sounds in bubbly liquids.[9] Now here is a problem with *Q*/*S*. When applied to soliton solutions of the KdV equation, *Q*/*S* becomes identically zero, because the tendency for harmonic energy transfer due to nonlinearity is exactly counterbalanced by dispersion. It would nonetheless be wrong to suggest from $Q/S = 0$ that there is no nonlinearity. On the other hand, *C*/*S*, which becomes a nonzero constant, can shed light on the intricate balance between nonlinearity and dispersion leading up to solitons.

4. CONCLUSIONS

As an attempt to demystify the Morfey-Howell nonlinearity indicator *Q*/*S*, this paper has described an impedance-based interpretation, in which *Q*/*S* is viewed as a nonlinearly-induced change in specific acoustic impedance. Also, a new nonlinearity indicator *C*/*S* has been introduced, which, together with *Q*/*S*, constitutes a complete set of nonlinearity indicators for finite-amplitude waves in liquids with arbitrary absorption and dispersion laws.

REFERENCES

[1]D. T. Blackstock, M. F. Hamilton, and A. D. Pierce, “Progressive waves in lossless and lossy fluids,” in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 4, pp. 132–133.

[2]J. A. Gallagher and D. K. McLaughlin, “Experiments on the non-linear characteristics of noise propagation from low and moderate Reynolds number supersonic jets,” in 7th Aeroacoustics Conference (1981), AIAA Paper No. 81-2041.

[3]K. L. Gee, R. J. Kenny, T. B. Neilsen, T. W. Jerome, C. M. Hobbs, and M. M. James, “Spectral and statistical analysis of noise from reusable solid rocket motors,” Proc. Mtgs. Acoust. 18, 040002 (2013).

[4]Y. C. Kim and E. J. Powers, “Digital bispectral analysis and its applications to nonlinear wave interactions,” IEEE Trans. Plasma Sci. **7**, 120–131 (1979).

[5]C. L. Morfey and G. P. Howell, “Nonlinear propagation of aircraft noise in the atmosphere,” AIAA J. **19**, 986–992 (1981).

[6]B. O. Reichman, K. L. Gee, T. B. Neilsen, and K. G. Miller, “Quantitative analysis of a frequency-domain nonlinearity indicator,” J. Acoust. Soc. Am. **139**, 2505–2513 (2016).

[7]M. F. Hamilton and C. L. Morfey, “Model equations,” in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 3, pp. 56–57.

[8]M. J. Ablowitz, *Nonlinear Dispersive Waves* (Cambridge University Press, Cambridge, 2011), Chap. 5.

[9]M. F. Hamilton, Yu. A. Il’inskii, and E. A. Zabolotskaya, “Dispersion,” in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock (Acoustical Society of America, New York, 2008), Chap. 5, pp. 167–174.