Level-educed Wavepacket Representation of Noise Radiation from a High-performance Military Aircraft

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An analytical wavepacket-based jet noise model has been applied for the first time to high-performance, military aircraft noise. Ground-based acoustical measurements were made of a tethered high-performance military aircraft with one engine cycling through four engine conditions. The resulting spectra have been decomposed into fine and large-scale similarity spectral components. The spatial distribution of the large-scale similarity spectrum decomposition provides the ability to extract level-based, data-educed wavenumber spectra and an estimation of the convective speed as a function of frequency and engine condition. The data-educed wavenumber spectra are compared with the axial wavenumber amplitude spectra associated with an analytical wavepacket ansatz. A simulated annealing algorithm is employed to minimize the difference between the analytical wavenumber amplitude spectra and the data-educed wavenumber spectra. The frequency-dependent wavepacket shapes obtained from the optimizations follow expected trends as a function of distance of contracting length as frequency increases but appear to extend for approximately the same number of wavelengths. The level-based, data-educed wavenumber spectra model the Mach wave radiation associated with the large-scale turbulent mixing noise. This wavepacket model is a step towards producing an equivalent source representation of noise from tactical gas turbine engines to guide future noise environment modeling efforts.

Nomenclature

\[ a_n = \text{stochastic random function} \]
\[ A = \text{normalization factor} \]
\[ A_j = \text{scaled normalization factor} \]
\[ A_n = \text{axial wavenumber spectrum amplitude} \]
\[ b_1 = \text{length scale of growth of axial wavepacket} \]
\[ b_2 = \text{length scale of decay of axial wavepacket} \]
\[ c = \text{ambient sound speed} \]
\[ D_j = \text{jet nozzle diameter} \]
\[ f = \text{frequency} \]
\[ F_n = \text{azimuthal mode contributions} \]
\[ E = \text{cost function} \]

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\( g_1 \) = rate of growth of axial wavepacket
\( g_2 \) = rate of decay of axial wavepacket
\( G_n \) = normalized axial eigenfunctions
\( k \) = acoustic wavenumber
\( k_x \) = axial wavenumber
\( k_r \) = radial wavenumber
\( L_{w,\text{far}} \) = far field levels due to a wavepacket
\( M_c \) = convective Mach number
\( n \) = azimuthal mode number
\( p \) = acoustic pressure
\( p_0 \) = wavepacket axial shape
\( \bar{p}_n \) = axial wavenumber spectrum
\( p_\text{ref} \) = reference pressure
\( p_w \) = pressure modeled by a wavepacket
\( p_m \) = pressure modeled by a monopole
\( \bar{p}_n \) = axial wavenumber spectrum
\( Q \) = monopole source strength
\( r_0 \) = radius of the nozzle
\( r \) = radius in cylindrical coordinates
\( R \) = radius in spherical coordinates
\( S \) = spectral density
\( St \) = Strouhal number
\( t \) = time
\( U_c \) = convective velocity
\( U_j \) = jet velocity
\( z \) = axial distance
\( \varepsilon \) = frequency-dependent amplitude
\( \phi \) = azimuthal angle
\( \theta \) = polar angle
\( \lambda \) = wavelength
\( \omega \) = angular frequency
\( p \) = acoustic pressure
\( \rho_j \) = jet density

I. Introduction

Wavepacket representations of jet noise strive to provide a model consistent with linear stability theory of the mean flow\(^1,2\) that incorporates features of the highly directional turbulent mixing noise.\(^3,4,5\) A wavepacket is a spatially extended source characterized by an axial amplitude distribution that grows, saturates and decays, an axial phase relationship that produces directional noise,\(^6\) and correlation lengths longer than the integral length scales of the turbulence.\(^7\) Wavepacket characteristics are found in turbulent region, hydrodynamic near field and acoustic far field.\(^3\) However, depending on the convective Mach number (relative to the ambient sound speed), a wavepacket describes either primarily evanescent near-field sound radiation (subsonic) or the highly directional Mach wave radiation (supersonic). The goal is to create a wavepacket model of jet noise in future noise environment modeling and noise reduction efforts.

Wavepacket investigations of jet noise have been conducted using measurements in the turbulent region, hydrodynamic near field, and acoustic far field of laboratory-scale jets. Beginning with Mollo-Christensen,\(^8\) wavepacket-like features have been observed in near-field pressure measurements. More recently, visualization methods have been used to estimate wavepacket properties associated with the turbulent flow, as e.g., in Ref. 9. Arrays of microphones in the hydrodynamic near field have been used to detect linear instability waves and connect...
them to near-field pressure wave packets.\textsuperscript{10,11,12,13} Methods for linking the wave packets educed from the near-field data to the far-field sound radiation have also shown promise.\textsuperscript{6,7,14}

Based on linear stability theory,\textsuperscript{1,2} techniques have been proposed to use measured far-field data to infer wave packets for the radiating portion of the pressure fluctuations.\textsuperscript{15} Morris\textsuperscript{16} and Papamoschou\textsuperscript{17,18} present two methods of obtaining wave packets with the same goal: for a single frequency, connect far-field measured spectral levels with a wave packet representation of the source radiated pressure projected on a near-field cylinder a short distance from the jet centerline. Morris\textsuperscript{16} showed how the spatial distribution of levels from the large-scale similarity spectra decomposition yield frequency-dependent axial wavenumber spectra for a range of jet velocities. Papamoschou proposed an analytical wave packet model that can be used to predict measured far-field levels. However, to match the sideline levels, the contribution of a monopole was added to the wave packet field. Because both methods begin with far-field noise, information is not available about the nonradiating, evanescent components of the turbulent pressure variations in the hydrodynamic near field. Nevertheless, the acoustic field contains information sufficient to obtain an equivalent wave packet model representative of the propagating noise.

The initial application of the wave packets ansatz to full-scale, military aircraft noise begins with the procedure described by Morris\textsuperscript{16} for obtaining the azimuthally-averaged, wavenumber spectral amplitudes. The investigation into a wave packet representation for full-scale military jet noise continues by using the data-educated wavenumber amplitude spectra to find convective velocities from the peaks in the wavenumber spectra and to find frequency-dependent analytical wave packets that model the large-scale turbulent mixing noise. The five-parameter analytical wave packet model described in Papamoschou\textsuperscript{13} has been employed. The Fourier transform of the model’s complex wavenumber amplitude yields the corresponding analytical axial wavenumber spectrum, which is compared to the data-educated wavenumber spectra in the optimization. To provide insights into the relative uncertainty in the parameter estimates obtained by the optimization, a study concerning how the wave packet parameters relate to features in the wavenumber spectrum is presented. Finally, the analytical axial wavenumber spectrum can be used to estimate the far field levels as a function of angle and sideline distance. This initial evaluation of the ability of the wave packet ansatz to yield the levels measured in the vicinity of an high-performance military aircraft indicates there is potential and points towards modifications in the wave packet model that will enhance the prediction.

II. Background

In this paper, two methodologies are linked together and expanded to create a frequency-dependent wave packet model for noise near a high-performance military aircraft. First, the measured spectra are decomposed into the two similarity spectra for turbulent mixing noise given by Tam \textit{et al.}\textsuperscript{21} Following the derivations in Tam and Chen,\textsuperscript{19} Tam,\textsuperscript{20} and Morris,\textsuperscript{16} the spatial dependence of levels associated with the large-scale similarity spectrum is called the measured spectral density and provides estimates of the axial wavenumber spectrum as a function of frequency. These data-educated spectra are compared to the spatial Fourier transforms of analytical wavepacket models given in Papamoschou.\textsuperscript{17} The corresponding wavenumber spectra provide numerical estimates of the far-field noise levels, again following the derivation in Papamoschou.\textsuperscript{17} These theoretical developments are presented in this section. The nomenclature has been modified slightly from that originally presented to provide a consistent framework tying the methods together, which helps with the physical interpretation of the results.

Before beginning the derivations, the relationship between the acoustic, axial, and radial wavenumbers needs to be considered. For a sound wave of frequency $\omega$ traveling in a medium with wave speed $c_o$, the magnitude of the acoustic wavenumber is $k = \omega/c_o$, and the wavenumber vector, $\mathbf{k}$, points in the direction of the traveling in traveling. In cylindrical coordinates, the $z$ and $r$ components of the acoustic wavenumber vector are $k_z = k \sin \theta$ and $k_r = k \cos \theta$, such that $k^2 = k_z^2 + k_r^2$. The angle at which the wavenumber vector points relative to the $z$ axis is $\theta = \tan^{-1}(k_r/k_z)$. The wavenumber associated with radial direction, $k_r = \pm \sqrt{k^2 - k_z^2}$, is real if $|k_z| \leq \omega/c_o$ and in such cases corresponds with waves propagating radially outward from the source. Such wavenumbers are called sonic (when equal) or supersonic, signifying trace wavenumber matching in the axial direction. Note that $k_r = \pm \sqrt{k^2 - k_z^2}$ is imaginary when $k_z > k$. The positive or negative sign associated with the square root is chosen such that when $k_z > k$ the radial components decay exponentially. For subsonic wavenumbers, the associated wavenumber components decay evanescently with radial distance. Because our measurements are not in the hydrodynamic near field (more than a half wavelength away at frequencies of interest), the $k_r$ values represented in the axial wavenumber spectra are restricted to propagating waves with supersonic axial wavenumbers: $|k_z| \leq \omega/c_o$.

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that where radius dependent fluctuations on a cylindrical surface coaxial with the jet to be broadband and nondeterministic. The large-scale similarity (LSS) spectrum approximates the spectral shape in the maximum sound radiation direction, often referred to as the Mach cone. In between these two regions, a combination of the FSS and LSS spectra is needed to account for the spectral shape. Examples of experimental implementation of the decomposition of measured jet noise spectra into FSS and LSS components are found in Morris, Tam et al., and Viswanathan, for laboratory-scale jets and for military aircraft engines in Schlinker et al. and Neilsen et al. When this spectral decomposition is performed for an array of microphones, the spatial variation in the levels associated with the LSS component, referred to as the LSS spectral density, can be tied to the wavepacket ansatz.

The connection between wavepacket amplitudes and the measured spectral density presented in this subsection follows closely that presented in Tam and Chen, Tam, and Morris. First, the wave equation for pressure fluctuations from a wavepacket is represented as a normalized eigenfunction expansion. Second, an expression for the coaxial pressure fluctuations on a cylindrical surface due to stochastic jet noise leads to an expression for the autocorrelation. The ensemble average of the autocorrelation of these pressure fluctuations can be related to those of the noise exiting the nozzle due to the self-similar nature of the turbulence. Each step is now explained in detail.

The three-dimensional wave equations in cylindrical coordinates is

\[
\frac{\partial^2 p}{\partial t^2} - c_0^2 \left( \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2} \right) = 0.
\]

The general solution in cylindrical coordinates, \(p(r, \phi, z, t)\), can be represented in terms of its Fourier transform with respect to time, \(t\), and axial distance, \(z\), as well as a Fourier series in the azimuthal angle \(\phi\):

\[
p(r, \phi, z, t) = \frac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_n(r, k_z, \omega) \exp[i(n\phi + k_z z - \omega t)] \, dw \, dk_z,
\]

with \(\omega\) as the radian frequency and \(k_z\) as the axial wavenumber. From the wave equation, the corresponding ordinary differential equation for \(P_n(r, k_z, \omega)\), the radially dependent wavenumber/frequency spectrum, is

\[
\frac{d^2 P_n}{dr^2} + \frac{1}{r} \frac{d P_n}{dr} + \left( \omega^2 - k_z^2 - \frac{n^2}{r^2} \right) P_n = 0.
\]

The general solution is

\[
P_n(r, k_z, \omega) = A_n(k_z, \omega) H_n^{(1)}(k_r, r),
\]

where \(H_n^{(1)}(\xi)\) is the Hankel function of the first kind of order \(n\) and argument \(\xi\), and \(k_r\) is the radial wavenumber. The amplitudes \(\{A_n(k_z, \omega)\}\) constitute the variations of the acoustic pressure as a function of axial wavenumber and frequency on a cylinder a distance \(r\) from the jet centerline.

From previous work, it was found that the random or stochastic nature of jet noise causes the pressure fluctuations on a cylindrical surface coaxial with the jet to be broadband and nondeterministic. The frequency-dependent wavenumber spectrum of the pressure fluctuations of the \(n\)th azimuthal mode on a cylindrical surface of radius \(r_0\) can be expressed in terms of a modal series expansion as

\[
P_n(r_0, k_z, \omega) = a_n(\omega) G_n(k_z, \omega),
\]

where \(a_n(\omega)\) is a stochastic random function of frequency and \(G_n(k_z, \omega)\) are axial eigenfunctions normalized such that

A. Level-educed Wavenumber Spectrum

One model for jet noise postulates that sound radiated from the jet exhaust plume is generated by turbulent mixing noise from fine-scale and large-scale turbulent structures. Tam et al. used an extensive database of laboratory-scale jet data and found a similarity spectrum associated with each kind of turbulent mixing noise. To the sideline direction, the fine-scale similarity (FSS) spectrum matches the radiated noise spectrum for many jet operating conditions. The large-scale similarity (LSS) spectrum approximates the spectral shape in the maximum sound radiation direction, often referred to as the Mach cone. In between these two regions, a combination of the FSS and LSS spectra is needed to account for the spectral shape. Examples of experimental implementation of the decomposition of measured jet noise spectra into FSS and LSS components are found in Morris, Tam et al., and Viswanathan, for laboratory-scale jets and for military aircraft engines in Schlinker et al. and Neilsen et al. When this spectral decomposition is performed for an array of microphones, the spatial variation in the levels associated with the LSS component, referred to as the LSS spectral density, can be tied to the wavepacket ansatz.

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\[
p(r, \phi, z, t) = \frac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_n(r, k_z, \omega) \exp[i(n\phi + k_z z - \omega t)] \, dw \, dk_z,
\]

with \(\omega\) as the radian frequency and \(k_z\) as the axial wavenumber. From the wave equation, the corresponding ordinary differential equation for \(P_n(r, k_z, \omega)\), the radially dependent wavenumber/frequency spectrum, is

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\]

The general solution is

\[
P_n(r, k_z, \omega) = A_n(k_z, \omega) H_n^{(1)}(k_r, r),
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\[ \int_{-\infty}^{\infty} G_n(k_x, \omega) dk_x = 1. \]

Equation (2) evaluated at \( r = r_o \) is equal to Eq. (3), which provides an expression for the amplitudes \( \{A_n(k_x, \omega)\} \), such that the frequency-dependent, wavenumber spectrum of the pressure fluctuations is

\[ P_n(r, k_x, \omega) = \frac{a_n(\omega)H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r r_0)} G_n(k_x, \omega). \]  \( \text{(4)} \)

The pressure fluctuations for \( r \geq r_o \) is obtained by taking the inverse Fourier transforms of Eq. (4) with respect to \( \omega \) and \( k_x \):

\[ p(r, \phi, z, t) = \frac{1}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( a_n(\omega)a_{n'}(\omega') \right) H_n^{(1)}(k_r r) H_{n'}^{(1)}(k_r' r) \right) G_n(k_x, \omega) \exp[i(n\phi + k_z z - \omega t)] d\omega dk_x. \]  \( \text{(5)} \)

Based on Eq. (5), the ensemble average of the autocorrelation of these pressure fluctuations is

\[ \langle p(r, \phi, z, t)p(r, \phi, z, t + \tau) \rangle = \frac{1}{(2\pi)^6} \sum_{n=-\infty}^{\infty} \sum_{n'}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \left( a_n(\omega)a_{n'}(\omega') \right) \right) H_n^{(1)}(k_r r) H_{n'}^{(1)}(k_r' r) \left) G_n(k_x, \omega) G_{n'}(k_x', \omega) \exp[i((n - n')\phi + (k_z - k_z') z - (\omega - \omega') t)] d\omega dk_x dk_x'. \]

This complex expression for the ensemble-averaged autocorrelation relates to the autocorrelation of the noise exiting the nozzle.

In Refs. [16, 19-20], the argument is made that since the instability waves associated with the large turbulent structures have no intrinsic characteristic time and length scales, they are self-similar. This is equivalent to saying that they are excited by white noise at the nozzle exit plane, \( z = 0 \), in which case, the autocorrelation functions contain delta functions in both polar angle and time. Using the notation of Morris,\(^{16}\) this yields

\[ \left( \frac{\langle p(r, \phi, 0, t)p(r, \phi + \chi, 0, t + \tau) \rangle}{\rho_j U_j^2} \right) = A^2 \rho_j^2 U_j^2 D_j \delta(\chi) \delta(\tau). \]

where \( D_j, \rho_j \) and \( U_j \) are the diameter, density and speed of the jet, and \( A^2 \) is a normalization factor. This expression of the autocorrelation due to the self-similar nature of the turbulence can be connected to the stochastic property of the function \( a_n(\omega) \) by repeated use of the Fourier transform and application of the normalization such that

\[ \langle a_n(\omega)a_{n'}(\omega') \rangle = A^2 \rho_j^2 U_j^2 D_j \delta(\omega + \omega') \delta_{n,-n'} \]

where \( \langle \ast \rangle \) denotes the ensemble average, \( \delta_{n,-n'} \) is the Kronecker delta (which equals 1 when \( n = -n' \) and equals zero otherwise) and \( \delta(\omega + \omega') \) is the Dirac delta function:

\[ \delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(\omega - \omega')} dp. \]

The sifting properties of the Delta functions and the relationship between the positive and negative sides of the Fourier transforms lead to a simpler expression for the autocorrelation at position \( (r, \phi, z) \) relative to pressure fluctuations on a cylinder of radius \( r_o \), namely

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\begin{align*}
(p(r, \phi, z, t)p(r, \phi, z, t + \tau)) &= \frac{A^2}{2\pi \rho^2 \mu^3 D} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |F_n(r, z, \omega)|^2 \exp(-i\omega \tau) d\omega,
\end{align*}

where
\begin{equation}
F_n(r, z, \omega) = \int_{-\infty}^{\infty} G_n(k_z, \omega) \frac{H_n^{(1)}(k_z r)}{H_n^{(3)}(k_z r_0)} \exp(ik_z z) dk_z.
\end{equation}

Because the Fourier transform with respect to time of the autocorrelation is the autospectral density,
\begin{equation}
S(r, z, \omega) = \int_{-\infty}^{\infty} (p(r, \phi, z, t)p(r, \phi, z, t + \tau)) \exp(i\omega \tau) d\tau,
\end{equation}

which can be expressed as a sum of the modal contributions, \(F_n(r, z, \omega)\) in Eq. (6), by
\begin{equation}
S(r, z, \omega) = \frac{A^2}{2\pi \rho^2 \mu^3 D} \sum_{n=-\infty}^{\infty} |F_n(r, z, \omega)|^2.
\end{equation}

Thus, the spectral density at a location \((r, z)\) and angular frequency, \(\omega\), may be expressed as the sum over azimuthal modes of the inverse Fourier transform with respect to \(k_z\) of the axial wavenumber spectrum (in Eq.(4)) on a cylindrical surface of radius \(r\).

Although there is a summation over \(n\) in Eq. (7), use of only the \(n = 0\) term is often a good approximation, especially for lower frequency noise from axisymmetric jets.16,20,22 Restriction to only the \(n = 0\) term is also applied if the data span a limited azimuthal aperture so as to lack sufficient information to estimate the contributions from higher-order azimuthal modes \((n > 0)\). In this case, an azimuthally averaged result is obtained. Thus, the measured spectral density, \(S(r, z, \omega)\), can be used to find the scaled magnitude of \(G_0(k_z, \omega)\), the axial wavenumber spectra of order zero.

The connection between \(G_0(k_z, \omega)\) and \(S(r, z, \omega)\) is more straightforward if a transformation is made from cylindrical to spherical coordinates using \(z = R \cos(\theta)\) and \(r = R \sin(\theta)\), where \(R\) is the distance from the origin to the point at \((r, z)\). In addition, the Hankel function can be approximated for large arguments by its asymptotic form:
\begin{equation}
H_0^{(1)}(\zeta) \to \frac{2}{\sqrt{\pi \zeta}} \exp \left[i \left(\zeta - \frac{\pi}{4}\right)\right] \text{as } |\zeta| \to \infty.
\end{equation}

With these modifications, the integral for \(F_0(R, \theta, \omega)\) in Eq. (6) is evaluated by the method of stationary phase to obtain
\begin{equation}
F_0(r, \theta, \omega) = -\frac{2i}{R} \frac{G_0(k_z, \omega)}{H_0^{(3)}(k_z r_0)} \frac{\omega R}{c_v} e^{i \omega \tau}.
\end{equation}

Inserting the expression in Eq. (9) into Eq. (7) yields the relationship
\begin{equation}
S(R, \theta, \omega) = \frac{A^2}{2\pi \rho^2 \mu^3 D} \left| \frac{1}{R} \frac{G_0(k_z, \omega)}{H_0^{(3)}(k_z r_0)} \right|^2.
\end{equation}

This expression can be rewritten as
\[
\frac{A^2 \rho^2 U_j^3 |G_0(k_x, \omega)|^2}{D_j^2} = \frac{\pi}{2} \left( \frac{R}{D_j} \right)^2 |H_0^1(k_r r_0)|^2 \frac{S(R, \theta, \omega)}{D_j}. \tag{10}
\]

The quantities on the right-hand side of Eq. (10), \(R, D_j, r_0, S(R, \theta, \omega)\), and \(k_r = k \sin \theta\) are measured quantities. The left-hand side is the squared amplitude axial wavenumber spectrum associated with the \(n = 0\) azimuthal mode scaled by the jet operating parameters. In the case of installed engines, the jet operating parameters \(\rho_j\) and \(U_j\) are not available. Thus, we rewrite Eq. (10) as

\[
\frac{A_j^2}{D_j^2} |G_0(k_x, \omega)|^2 = \frac{\pi}{2} \left( \frac{R}{D_j} \right)^2 |H_0^1(k_r r_0)|^2 \frac{S(R, \theta, \omega)}{D_j}, \tag{11}
\]

where \(A_j = A \rho_j U_j^{3/2}\) depends on the jet operating conditions. This expression for the scaled axial wavenumber spectrum, \(G_0(k_x, \omega)\), from the measurement-derived LSS spectral amplitudes in \(S(R, \theta, \omega)\) corresponds to data-educed estimates of the spatial Fourier transform of the wavepacket associated with the \(n = 0\) azimuthal mode on a cylindrical surface concentric with the nozzle exit.

### B. Analytical Wavepacket Model

One approach to wavepacket modeling of jet noise is to define an analytical wavepacket shape, and its wavenumber spectrum, that can be used to predict far field levels of the radiated field. The first step is to define a pressure fluctuation, referred to as a wavepacket, on cylindrical surface at \(r = r_0\), of frequency \(\omega\) and azimuthal mode number \(n\), as

\[
p_w(n, r_0, z, \phi, t) = p_0(z)e^{-i\omega t + im\phi}, \tag{12}
\]

where \(\phi\) is the azimuthal angle. The wavepacket axial shape is \(p_0(z)\) is composed of an amplification-decay amplitude envelope and an oscillating portion: \(p_0(z) = |p_0(z)|e^{iaz}\), where \(\alpha = \omega/U_c\) is the convective wavenumber for convective speed \(U_c\).

Although there are many options for the amplitude envelope, the wavepacket axial shape defined by Papamoschou in Eq. [35] of Ref. [17], is implemented in our initial investigation into the ability of a wavepacket to represent the turbulent mixing noise associated with the large-scale turbulent structures. The candidate wavepacket model is

\[
p_0(z) = \tanh \left( \left( \frac{z}{b_1} \right)^{\beta_1} \right) \left\{ 1 - \tanh \left( \left( \frac{z}{b_2} \right)^{\beta_2} \right) \right\} e^{iaz}. \tag{13}
\]

The parameters of the first hyperbolic tangent term, \(b_1\) and \(g_1\), control the length scale and the rate of the growth of the wavepacket amplitude. Similarly, \(b_2\) and \(g_2\) dictate the length scale and rate of the decay. Some examples of the real part of \(p_0\) for different combinations of the four parameters \([b_1, b_2, g_1, g_2]\) are displayed Figure 1 to illustrate the effect they have on the axial amplitude distribution. Each wavepacket is normalized such that the maximum value of \(|p_0(z)|\) is unity in each case. As the wavepackets in the first row of Figure 1 has different values of \(b_1\), the location of the first oscillation moves to larger \(z\) as \(b_1\) increases up to the value of \(b_2\). However, once \(b_1 > b_2\), the shape of the wavepacket does not appear to change significantly. On the second row, is varied such that an increase in the parameter delays the onset of the oscillations and, in effect, shortens the wavepacket. Similarly, an increase in \(g_2\) shortens the decay and thus the extent. The bottom row of wavepackets in Figure 1 illustrates how the wavepacket shape is related to the convective wavenumber, \(\alpha\), for three values of the convective speed \(U_c\): an increase in the wavenumber produces additional oscillations within in the wavepacket. These parameters provide the capability of finely adjusting the wavepacket model, and an understanding of their impact to the overall shape is necessary to interpret comparisons.

Each axial wavepacket shape has a corresponding wavenumber spectrum: \(\hat{p}_0(k_x) = \mathcal{F} \{ p_0(z) \} \). Normalized wavenumber spectra for the cases in Figure 1 are displayed in Figure 2. Larger values of \(b_1\) and \(b_2\) create a wider
wavepacket and thus a narrower spectrum. However, there is not a significant change in spectral shape for cases with $b_1 > b_2$. For larger values of $g_2$, there is a bend in the spectrum, except when $g_1 > g_2$. In addition, a low convective speed, and hence a large convective wavenumber, has a much broader spectrum (red line) than the others. While not a comprehensive list of how the parameters affect the spectrum, the examples provided highlight how features of the wavenumber spectrum can be tied to the wavepacket model. These analytical wavenumber spectra can be compared to data-derived ones to obtain a wavepacket model for full-scale jet noise. In practice, modeling parameters are found by a simulated annealing algorithm based on minimizing the least-squares error between the two spectra. This is described further in Section II.D.

![Figure 1 Examples of the real parts of the axial wavepacket shapes at 250 Hz for the parameters listed above each plot: $[U_c, b_1, b_2, g_1, g_2]$.](image1)

![Figure 2 Wavenumber spectra of the wavepacket shapes shown in Figure 1. The modeling parameters are listed above each plot: $[U_c, b_1, b_2, g_1, g_2]$.](image2)
C. Far-field levels

Once an estimated wavenumber spectrum is obtained at each frequency of interest, both Morris\textsuperscript{16} and Papmoschou\textsuperscript{17} describe the connection between the wavenumber spectrum and modeling the far field sound pressure levels. Beginning with Eq. (5), the pressure, for \( r \geq r_0 \), is given by the multi-dimensional inverse Fourier transform of the wavenumber spectrum associated with a single angular frequency \( \omega \):

\[
p_w(n, r, \phi, z, t) = \frac{1}{2\pi} e^{-i\omega t + in\phi} \int_{-\infty}^{\infty} \frac{\hat{p}_0(k_z)}{H_n^{(1)}(k, r)} e^{ik_z z} dk_z.
\]

This integral can be divided into two parts corresponding with subsonic and supersonic wavenumbers:

\[
\begin{align*}
p_{w,\text{sub}}(n, r, \phi, z, t) &= \frac{1}{2\pi} e^{-i\omega t + in\phi} \int_{|k_z| > \omega/c_0} \frac{\hat{p}_0(k_z)}{H_n^{(1)}(k, r)} e^{ik_z z} dk_z, \\
p_{w,\text{sup}}(n, r, \phi, z, t) &= \frac{1}{2\pi} e^{-i\omega t + in\phi} \int_{-\omega/c_0}^{\omega/c_0} \frac{\hat{p}_0(k_z)}{H_n^{(1)}(k, r)} e^{ik_z z} dk_z \\
&= \frac{1}{\pi R} \frac{\epsilon(\omega) U_c}{\omega} P_0 \left( \frac{k_z U_c}{\omega} \right) e^{ikr e^{-i\omega t + i\phi}},
\end{align*}
\]

(14)

The far-field approximation of Eq. (14) yields an expression in spherical coordinates for the pressure at far-field position \((R, \theta, \phi)\) due to the supersonic portions of the pressure fluctuation that propagate to the far field:

\[
p_{w,\text{far}}(n, R, \theta, \phi, t) = \frac{i}{\pi R} \frac{\epsilon(\omega) U_c}{\omega} P_0 \left( \frac{k_z U_c}{\omega} \right) e^{ikr e^{-i\omega t + i\phi}},
\]

(15)

where \( R \) is the distance of the observer from the origin, and \( \theta \) is the angle from the wavepacket (jet) axis, \( z \).

When using the analytical wavepacket model, such as those in Papamoschou\textsuperscript{17-18}, it is convenient to represent the axial dependence of the amplitude of the wavepacket in a self-similar form:

\[
\hat{p}_0(k_z) = \frac{\epsilon(\omega) U_c}{\omega} P_0 \left( \frac{k_z U_c}{\omega} \right).
\]

With this assumption, the far field pressure from the wavepacket becomes

\[
p_{w,\text{far}}(n, R, \theta, \phi, \omega, t) = -\frac{i}{\pi R} \frac{\epsilon(\omega) U_c}{\omega} P_0 \left( \frac{k_z U_c}{\omega} \right) e^{ikr e^{-i\omega t + i\phi}},
\]

with all the \( \omega \) dependence explicitly shown. This can be simplified to

\[
p_{w,\text{far}}(n, R, \theta, \phi, \omega, t) = -\frac{i}{\pi R} \frac{\epsilon(\omega) U_c}{\omega} P_0 \left( \frac{M_c \cos \theta}{c_0 \omega r_0 \sin \theta} \right) e^{ikr e^{-i\omega t + i\phi}},
\]

because \( k_r = k \sin \theta \), \( k_z = k \cos \theta \), and \( k U_c / \omega = U_c / c_0 = M_c \). The modulus of the squared pressure can be expressed as

\[
S_{w,\text{far}}(n, R, \theta, \phi, \omega) = \left( \frac{\epsilon(\omega) U_c}{\pi R \omega} \right)^2 \left( \frac{M_c \cos \theta}{c_0 \omega r_0 \sin \theta} \right)^2.
\]

(16)

The sound pressure level, in decibels, predicted at this far-field location due to the wavepacket is
\[
L_{w,\text{far}}(n, R, \theta, \phi) = 10 \log \left( \frac{\left| S_{w,\text{far}}(n, R, \theta, \phi) \right|^2}{p_{\text{ref}}^2} \right)
\]  

(17)

where \(p_{\text{ref}} = 20 \mu \text{Pa}\). These predicted levels can be compared to measured levels to evaluate the validity of a wavepacket model for jet noise.

Although this representation of the far field due to the wavepacket is a function of angle from the jet axis, Papamoschou,\(^\text{17}\) found it necessary to add a monopole to capture the directivity of the jet noise to the sideline of the nozzle exit (large polar angles). The monopole at the origin produces a field \(p_m(R, t) = \frac{Q}{R} e^{-i \omega t + ikR}\). Combining this monopole and the far-field pressure from the wavepacket yields square pressure amplitudes of

\[
S_{\text{mod}} = S_{w,\text{far}} + \left( \frac{Q}{R} \right)^2.
\]

The process described in this section to obtain analytical wavepacket model for jet noise from the measured spectral density and predict the far field radiation is now applied to the measured noise.

D. Analytical Wavepacket Eduction

The overall goal of this work is to find frequency-dependent, analytical wavenumber spectra that match the data-duced wavenumber spectra via optimization. A simulated annealing algorithm is used to find modeling parameters that minimize the difference in the wavenumber spectra. However, because not all parameters are equally important in determining the quality of the match, a sensitivity study highlights the manner in which the analytical wavenumber spectrum shapes relates to changes in the modeling parameters. This is illustrated in Figure 3 for a specific example of a data-duced wavenumber spectrum. The first step in this comparison is to use the wavenumber that corresponds to the peak in the data-derived spectrum to estimate convective speed \(U_c\), and consequently \(a\), in Eq. (13). The correct value of \(U_c\) is essential to obtaining the same peak in the modeled spectrum. Thus, the \(U_c\) is most closely tied to the quality of the agreement between the wavenumber spectra, and the estimate of \(U_c\) obtained by the optimization has the smallest uncertainty of the five modeling parameters. To highlight the effect of the remaining four parameters on the shape of the wavenumber spectrum, each parameter is adjusted in turn. The value of parameter \(b_2\) is most closely tied to the width of the wavenumber spectrum near the peak. This is illustrated in the upper left plot of Figure 3, in which \(b_2\) is varied while the other parameters are held fixed. From this, it appears that a value of \(b_2 \approx 6\) gives approximately the curvature of the example wavenumber spectrum near its peak. The value of \(b_2\) is varied while \(b_1\) is varied in the upper right plot. Changes in \(b_1\) do not dramatically affect the shape of the wavenumber spectrum near the peak but appear, for low values, to introduce a bend. For values of \(b_1 > b_2\), the wavenumber spectrum does not change. A value of \(b_1 = 1.5\) yields an analytical wavenumber spectrum that most closely matches this example spectrum. The lower left plots in Figure 3 shows that while very low values of \(g_2\) produce a narrower wavenumber spectrum, the width remains the same near the peak for \(g_2\) larger than about 2. Larger values of \(g_2\) also introduce a bend in the corresponding spectra. The lower right plots in Figure 3 shows \(g_1\) (right) also do not significantly change the width of the modeled wavenumber spectra near the peak, but higher values of \(g_1\) widen the overall extent of the spectrum. This sensitivity study provides insight into the relative uncertainty of wavepacket modeling parameters obtained from an optimization designed to minimize the difference between data-duced and modeled wavenumber spectra.

Due to the uncertainty inherent in the inverse processing, it is useful to evaluate how the analytical wavepacket shape changes if the modeling parameters \(b_1, b_2, g_1, \) and \(g_2\) are adjusted 10% higher or lower. The real parts of the resulting wavepackets are displayed in Figure 4. An increase in the amplification parameters \(b_1\) and \(g_1\) increase the height of the initial oscillations, whereas lower the decay parameters \(b_2\) and \(g_2\) decreases the height of the later oscillations. The spatial rate of oscillation of the real part of the wavepacket is controlled by the convective speed, which is related to the peak in the wavenumber spectrum, which is not varied in this example.

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Figure 3 Examples of wavenumber spectrum for the parameters indicated in each plot. The parameter with the value indicated by “…” in the upper text box is the one whose values are listed in the legend.
III. Discussion

The wavenumber education method described in Refs. 1, 2, and 16 has been applied to noise measured in the vicinity of a tethered, high-performance military aircraft. This methodology has yielded, for the first time, the amplitudes of the axial wavenumber spectra for a wavepacket representation of full-scale military jet noise. The general features in the resulting data-educed wavenumber spectra are similar to those found for laboratory-scale jet in Ref. 16: (1) the peak in the wavenumber spectrum is not present at intermediate engine power or lower jet velocities, indicative of low Mach wave radiation, and (2) the wavenumber associated with the peak in the wavenumber spectrum increases with frequency, which corresponds directly to the changes in angle at which the coherent radiation is being emitted and the corresponding convective speed. In addition, a broader wavenumber spectrum at higher frequencies potentially implies a more compact wavepacket source region. Another alternative is that the data-educated wavenumber spectra contain the superposition of multiple wavepackets, which is the subject of an ongoing study.

Simulated annealing optimizations have been employed to minimize the difference between the data-educed wavenumber spectra for military power and the axial wavenumber spectra associated with the analytical wavepacket model from Papamoschou. The corresponding frequency-dependent wavepacket shapes show the expected source contraction with increasing frequency when plotted vs. axial distance. However, when the axial distance is scaled by wavelength, the spatial extent of the wavepackets is not substantially different for band center frequencies spanning three octaves. The self-similar nature of the optimized axial wavepackets has not been reported previously.

The optimized wavepackets can predict the measured field with three exceptions: the dual-lobe in the maximum radiation direction, the sideline radiation, and the nonlinear steepening, none of which are included in the LSS decomposition used to obtain the data-educed wavenumber spectra. Studies into the nature of the dual spatial lobe and corresponding dual spectral peak are ongoing. It is possible that the summation of mutually incoherent wavepackets could eventually model this dual-lobe phenomena. Additional efforts are needed to construct an equivalent source model that better predicts measured far-field levels to the sideline. With regard to improving the sideline match, Papamoschou in Ref. 17 added a monopole source to $S_{S_{\text{far}}}$, which essentially adds uncorrelated noise to the model. The possibility of creating an equivalent source model that incorporates both a correlated and uncorrelated source distribution was explored in Morgan et al. In that case, two line arrays with Rayleigh distributed amplitudes were used: the point sources on one array have a constant phase relationship derived from the far-field directivity angle, while the others have random phase relationship. Another idea is to incorporate the omnidirectional nature of the sideline radiation by including a Gaussian-based wavepacket-like model. The Gaussian parameter could be selected to fit a wavenumber decomposition of the FSS contributions to the overall levels, similar to as was done here, and in Ref. 16, with the LSS contributions. Alternately, it is possible that the
wavenumber spectrum of the measured levels could be used to define wavepackets models for the jet noise, instead of doing the similarity spectra decomposition.

Moving forward, a different analytical model of the axial wavepacket is going to be selected. There are two main reasons to change the model. First, of the five modeling parameters, only two, \( U_x \) and \( b_2 \), control the primary features of the axial wavenumber spectrum. Thus, the uncertainty in the estimates of the remaining three parameters is very large. A model with fewer parameters all of which impact the shape of the wavenumber spectrum is a better candidate. Second, the analytical wavepacket shape defined in Eq. (5) goes to zero when the distance is zero and increases on both sides. As a result, if the origin of the wavepacket is set to the reference point at \( z = 5.5 \) m, the resulting wavepacket envelope is bimodal, which is not physical. Because of this limitation, a second set of data-educed wavenumber spectra are calculated for the reference point at \( z = 0 \) for the wavenumber spectra and used in the optimizations. This difference in reference point and definition of angle is significant near the jet but decreases in importance as the distance to the measurements increases. A new wavepacket model that is continuous as the axial distance becomes negative is advantageous.

There are limitations in using the level-based analyses to construct an equivalent source for jet noise. First, because only the magnitudes of the data-educed wavenumber are obtained from the LSS spectral density, the location of the peak in the axial wavepackets is not well defined. Second, a single, coherent wavepacket cannot reproduce the complex coherence of the measured field, as explained in Harker et al. A more complete equivalent source model needs to account for the field coherence as well as the measured levels. Such a model is needed to successfully model the noise environment of a high performance military aircraft.

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References