Collisional Processes at Low Densities in Magnetic Mirror Systems

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(Received 12 October 1970; final manuscript received 31 March 1971)

The study of collisional processes in plasmas produced by neutral-atom injection into magnetic mirror fields is described. The emphasis is on the many collisional processes which occur as the plasma density increases. Experimental and theoretical results are given. The experimental results are discussed first in terms of a simple model which assumes a Maxwellian electron distribution and a monoenergetic ion component of much higher energy. Analytical solutions may be obtained for this model. Also presented is a more complete theory employing two time-dependent Fokker–Planck equations to describe the behavior of the electron and ion distribution functions. Both models are in good agreement with measured values of the electron temperature and plasma potential. The equilibrium values of these two quantities are found to vary as the $\frac{1}{3}$ power of the ratio of the plasma density to the background-gas density.

I. INTRODUCTION

In plasmas contained by "open-ended" or magnetic mirror systems, particles may be lost from the system by collisions that scatter the particles into the mirror loss cones. Chandrasekhar\textsuperscript{1} has considered similar scattering effects for stellar collisions. Spitzer,\textsuperscript{2} by extending Chandrasekhar's theory to an ionized plasma, has shown that the characteristic time for loss of a charged particle by scattering is proportional to the $\frac{1}{3}$ power of the particle energy. Therefore, unless the charge species of the plasma have the same energy, the scattering loss times generally will be different; and an electrostatic potential then develops to balance the loss rates, resulting in approximate charge neutrality.

In magnetic mirror systems where the ion energy is large compared with the mean electron energy a positive plasma potential develops, retarding the electron loss but increasing the ion loss. In an early investigation, Kaufman\textsuperscript{3} assumed that the magnetic field configuration could be approximated by a "square well." He found that electrons either were confined or had their loss cones reduced by a positive potential. For ions, however, the loss cone was increased; and a minimum energy was found to exist for which there was no confinement due to expulsion by the positive potential. Other authors\textsuperscript{4–7} have included a variation of the electrostatic potential along the direction of the magnetic field lines, instead of starting from Kaufman's "square-well" assumption.

To describe the electron temperature and electrostatic potential of a plasma produced by neutral-atom injection into a magnetic mirror field, additional source and loss terms are necessary. A simple model including these extra terms was presented in an earlier paper by three of the present authors.\textsuperscript{8} This model, which assumes a Maxwellian electron distribution and a monoenergetic ion component of a much higher energy, employs two buildup equations (first-order differential equations) to describe the collisional processes affecting the ion and electron densities. The success of this approach in describing the measured electron temperature and plasma potential has led to refinement of the original model to include, for the previous approximation $\exp(-e\phi/kT_e)$, the substitution of an exact expression for the fraction of electrons that have energy greater than $e\phi$. In addition, the ion scattering term is not neglected now, and the initial electron energy is assumed to have a value other than zero.

If the distribution function for a particular plasma species is expected to change in an unknown way, then the ordinary differential equation describing the density of that species must be replaced by a partial differential equation describing the distribution function of that species in velocity space. The most suitable mathematical description comes from the Fokker–Planck equation. This is because the dominant mechanism for energy transfer among the particles is by long-range Coulomb interactions. The Fokker–Planck equation for the distribution functions of several species of particles, where the two-body force is an inverse square law, has been derived by Rosenbluth, MacDonald, and Judd.\textsuperscript{9} This equation has been used by many authors for numerical calculations of end-loss rates.\textsuperscript{7,10–14}

In the present work, the Fokker–Planck equation serves as a tool for understanding the time dependence of plasma parameters such as density, electron temperature, electrostatic potential, and the distribution functions. Two time-dependent Fokker–Planck equations are employed to describe the behavior of the electron and ion distribution functions. The Fokker–Planck equations and the method for obtaining numerical solutions have previously been described.\textsuperscript{15}

The theory describing the collisional processes in a plasma produced by neutral-atom injection in a magnetic mirror field is described in Sec. II, where it is discussed in terms of buildup equations for the density of the ion and electron plasma components and also in terms of the corresponding Fokker–Planck equations for the distribution functions. In Sec. III, the experimental measurements of the plasma potential $\phi$ and the electron temperature $T_e$ are described. The plasma potential measurements were obtained by monitoring the parallel energy of the slow ions leaving the plasma.
along the magnetic axis. Relative measurements of the electron temperature were obtained from the amplitude of microwave radiation emitted at the electron cyclotron frequency.

In Sec. IV, we compare theoretical calculations and experimental measurements. Some additional results on collisional broadening of the ion distribution function are discussed in Sec. V. All experimental and theoretical results in this paper assume the plasma ions to be protons. Section VI contains some concluding remarks.

II. THEORY

A. Simple Model Employing First-Order Differential Equations

The electron temperature $T_e$ and plasma potential $\phi$ of a plasma produced by neutral-atom injection into a magnetic mirror field may be obtained by considering the buildup equations for the ion density $n_i$ and the electron density $n_e$ of the plasma. These equations, which assume a Maxwellian electron distribution and a monoenergetic ion component, describe the collisional processes in the plasma in terms of cross sections averaged over the relative velocities of the interacting particles.

The energetic neutral-atom beam $I$ is trapped by Lorentz ionization of excited neutral atoms produced by charge-exchange collisions in a gas or vapor. Other trapping processes include ionizing collisions with the background gas and with previously trapped ions and electrons. Cross sections for these processes are represented by $\sigma_{ie}$, $\sigma_{it}$, and $\sigma_{it}^s$, respectively. Cross sections for ion losses by charge-exchange collisions and by scattering into the mirror loss cone are represented by $\sigma_{es}$ and $\sigma_{es}^s$. The cross section for electrons lost by scattering into the loss cone is $\sigma_{es}^s$. Electrons are also formed by ionizing collisions of ion and electrons with the background gas; these collisions are described by the cross sections $\sigma_{ie}$ and $\sigma_{ie}^s$, respectively.

For a background-gas density $n_0$ the differential equations for the ion and electron density become

\[
\frac{dn_i}{dt} = \frac{IF}{V} + \frac{IL}{V} n_0 \left( \frac{\sigma_{ie}^s v}{v} \right)
+ n_i \left( \frac{IL}{V} \left( \sigma_{ie}^s v + \sigma_{it}^s v \right) - n_0 \sigma_{es} v \right) - n_i \sigma_{es}^s v,
\]

(1)

\[
\frac{dn_e}{dt} = \frac{IF}{V} + \frac{IL}{V} n_0 \left( \frac{\sigma_{ie}^s v}{v} \right)
+ n_e \left( \frac{IL}{V} \left( \sigma_{ie}^s v + \sigma_{it}^s v \right) + n_0 (\sigma_{ie}^s v + \sigma_{it}^s v) \right)
- n_e \sigma_{es}^s v F \left( \frac{\phi}{kT_e} \right),
\]

(2)

where $F(\phi/kT_e)$ is the fraction of electrons with energy greater than $\phi$. The beam velocity, the relative velocity between colliding particles, and the electron velocity are given by $v_i$, $v_e$, and $v$, respectively. The path length of the beam through the plasma volume $V$ is given by $L$, and the fraction of the beam that is ionized by the Lorentz force of the magnetic field is given by $f^s$. Throughout this paper, $n_0$ refers to the number of molecules per cubic centimeter. We assume the background gas to be hydrogen.

In Eqs. (1) and (2), we assumed quasineutrality (i.e., $n_i \approx n_e$) and neglected the charge exchange of the trapped ions with the neutral beam. This latter process, which replaces one hot ion by another, has no net effect on spatially independent equations such as Eqs. (1) and (2). It has been shown that charge exchange of the trapped ions with the neutral beam may be regarded as a diffusion of the ions across the magnetic field lines for a finite plasma. For a distribution function which is not monoenergetic, this process tends to sharpen the trapped distribution.

The rate at which electrons leave the plasma by Coulomb scattering into the loss cones may be estimated by saying that the only effect of the plasma potential $\phi$ is to eliminate the loss of electrons with energy less than $\phi$. The fraction of scattered electrons having sufficient energy to escape over the potential barrier is

\[
F \left( \frac{\phi}{kT_e} \right) = 1 + \frac{2}{\pi^{1/2}} \left( \frac{\phi}{kT_e} \right)^{1/2} \exp \left( -\frac{\phi}{kT_e} \right) - \Phi \left( \frac{\phi}{kT_e} \right),
\]

(3)

where $\Phi(\phi/kT_e)$ is the probability integral. Combining (3) with the scattering rate gives the last term of Eq. (2).

In the following analysis, we make the additional assumption that the plasma "growth rate" is slow compared with the collisional relaxation times of the electrons; i.e., the electron temperature and scattering rates are in quasiequilibrium with the existing plasma density. The electrical neutrality of a plasma implies that

\[
\frac{dn_i}{dt} \approx \frac{dn_e}{dt}.
\]

(4)

By equating Eqs. (1) and (2), we obtain

\[
-n_i n_0 \sigma_{es} v - n_i \sigma_{es}^s v = n_e n_0 (\sigma_{ie}^s v + \sigma_{it}^s v) - n_i n_0 \sigma_{es} v F(\phi/kT_e).
\]

(5)

Rearranging terms and equating $n_i$ to $n_e$, we have

\[
F \left( \frac{\phi}{kT_e} \right) = \frac{n_0 (\sigma_{ie}^s v + \sigma_{it}^s v)}{n_0 (\sigma_{ie}^s v)} + \frac{\sigma_{es}^s v}{\sigma_{es} v}.
\]

(6)

A plasma contained in a magnetic mirror field is expected to generate a positive electrostatic potential if the ion energy is greater than the electron energy. Since the positive potential restricts electron escape,
the electrons are held by the plasma until they receive sufficient energy from collisions with the energetic ions to overcome the plasma potential. The electrons may then escape by scattering into the mirror loss cone. Thus, the plasma potential may be estimated by dividing the average energy the electrons obtain before escaping the plasma by the electronic charge; \( \phi \) is approximately equal to the average energy of the escaping electrons.

If the ion energy \( W \) is much greater than \( kT_e \), the energy transfer rate from the protons to the electrons is

\[
dW = \frac{1.8 \times 10^{-13} n_i W}{(kT_e)^{3/2}},
\]

where the ion energy and \( kT_e \) are expressed in keV. From Eq. (5), the average lifetime of an electron is given by

\[
\tau_e = \left[ n_0 \left( \sigma_v^e + \sigma_{cv}^e + \sigma_{ce}^e \right) + n_i \sigma_v^e \right]^{-1}.
\]

Therefore, the average energy of the escaping electrons is

\[
\phi = \frac{dW}{dt} \tau_e + E_0,
\]

or

\[
\phi = \frac{1.8 \times 10^{-13} n_i W}{(kT_e)^{3/2}} \left[ n_0 \left( \sigma_v^e + \sigma_{cv}^e + \sigma_{ce}^e \right) + n_i \sigma_v^e \right] + E_0,
\]

where \( E_0 \) is the energy with which the electrons are born. Equations (6) and (10) may be solved by an iterative process for the two unknowns \( \phi \) and \( T_e \).

If \( F(\phi/kT_e) \) is approximated by

\[
\exp(-\phi/kT_e)
\]

and the initial electron energy \( E_0 \) is neglected, then the following closed-form expressions for \( kT_e \) and \( \phi \) are obtained:

\[
kT_e = \left( \frac{-1.8 \times 10^{-13} n_i W}{\left[ n_0 \left( \sigma_v^e + \sigma_{cv}^e + \sigma_{ce}^e \right) + n_i \sigma_v^e \right] \ln K(n_0, n_i)} \right)^{2/3},
\]

\[
\phi = -kT_e \ln K(n_0, n_i),
\]

where

\[
K(n_0, n_i) = \left( \frac{n_0 \left( \sigma_v^e + \sigma_{cv}^e + \sigma_{ce}^e \right)}{n_i \sigma_v^e} \right) + \frac{\sigma_v^e}{n_i \sigma_v^e}.
\]

The foregoing state that the electron temperature and electrostatic potential are slowly varying functions of the ion density, i.e.,

\[
T_e \propto (n_i/n_0)^{2/13}
\]

and

\[
\phi \propto (n_i/n_0)^{3/16}
\]

if we neglect the dependence given by the logarithmic factor.

### B. Fokker–Planck Equations

Replacing the two first-order differential equations of Sec. II A with time-dependent Fokker–Planck equations for the ion and electron distribution functions enables us to solve a wider range of problems. Of particular interest are problems with time dependence and those which involve a change in the ion distribution function. The calculated electron distribution functions were always found to be Maxwellian in nature for densities above \( 10^9 \) particles/cm\(^2\).

As noted, the Fokker–Planck equation for an inverse-square force is derived in the paper of Rosenbluth, MacDonald, and Judd. In Eq. (31) of that paper, the result is given in spherical polar coordinates in

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**Fig. 1.** Electron and ion distribution functions as a function of \( v/v_0 \) for various times. The initial electron energy is 10 eV, and the initial ion energy is 1 keV.
velocity space, assuming azimuthal symmetry. We further assume that the distribution functions are isotropic in velocity space; i.e., there is no θ dependence. The functions depend only on v, the magnitude of the velocity, and t, the time. We have also modified the Fokker–Planck equation to include source and loss terms, so that for both the electron and ion distribution functions the equation takes the form

\[
\frac{\partial f_v}{\partial t} = A \frac{\partial^2 f_v}{\partial v^2} + B \frac{\partial f_v}{\partial v} + Cf_v + D.
\] (17)

The ion-loss term includes both Coulomb scattering into the loss cone and charge exchange. Additional discussion of this equation is given in the Appendix.

![Fig. 2. Plasma parameters \(n_0\), \(E_-\), \(E_+\), and \(\phi\) as a function of time, for an injection energy of 15 keV. The beam is turned off at a time corresponding to approximately 8.5 (dimensionless units).](image)

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The numerical code we employed is capable of providing graphical output plus numerical data. Figure 1 shows typical ion and electron distributions as a function of the normalized velocity \(v/v_0\) for various times during the buildup process. In this case, a 1-keV neutral beam consisting of \(6.2 \times 10^{14}\) atoms/sec was assumed to have a Lorentz trapping fraction, \(f^*\), equal to \(4.2 \times 10^{-3}\). The plasma volume \(V\) was 7.5 liters. Parameters such as plasma density \((n_0)\), average electron energy \((E_-)\), average ion energy \((E_+)\), and the plasma potential \((\phi)\) also can be plotted as a function of time, as shown in Fig. 2. The results here are for a 15-keV plasma reaching equilibrium between trapping and loss processes at approximately \(5 \times 10^8\) ions/cm³.

**III. EXPERIMENTAL METHOD**

**A. Measurement of the Plasma Potential**

In the Livermore neutral-injection experiment Alice, the potential of the plasma was determined experi-

![Fig. 3. Drawing showing location of detectors in relation to the plasma and the coil used to create the magnetic mirrors. The coil is wound in the shape of the beam on a baseball. The two main processes for producing slow ions are shown schematically.](image)

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mentally by monitoring the parallel energy of the slow ions leaving the plasma along the magnetic axis. The parallel ion energy is a measure of the plasma potential because the slow ions, born in the plasma, are pushed along the magnetic field lines by the positive potential, thus gaining energy by an amount corresponding to the potential. We assume the perpendicular energy of these ions to be small compared with their final parallel energy.

These slow ions are produced mainly by two types of processes. The first is the charge exchange of the trapped fast ions on the background gas, shown sche-

![Fig. 4. Slow-ion amplitude versus retarding bias, as measured by the gridded slow-ion detector. The intercept labeled \(\phi\) is considered the plasma potential.](image)

Fig. 4. Slow-ion amplitude versus retarding bias, as measured by the gridded slow-ion detector. The intercept labeled \(\phi\) is considered the plasma potential.
In Sec. IV we compare the decay time of the plasma potential, after the neutral beam is turned off, with the decay time of the trapped-ion density. We monitor ion density by observing the rate at which fast charge-exchange neutral particles leave the plasma. The fast-atom detector is a secondary-electron-emission type, positioned as shown in Fig. 3.

**B. Electron Temperature Measurements**

Measurements of the electron cyclotron radiation emitted from the plasma were made using a radiometer similar to that of Dicke. For a tenuous plasma, the electron cyclotron radiation is proportional to the product of the electron density and the electron temperature. Since measurements of the plasma density are available from other detectors, relative measurements of the electron temperature may easily be obtained from the intensity of the electron–cyclotron radiation.

Figure 5 is a schematic diagram of the apparatus used for the present measurements in the range 10.2–11.6 GHz. A circular horn 15 cm in diameter receives circularly polarized radiation from the plasma. A polarizer converts this circularly polarized radiation into linearly polarized radiation, preventing the loss of signal in the transition to rectangular wave guide. A manual switch permits radiation from either the plasma or a noise source to enter a latching circulator capable of switching rates up to 10 kHz. This device (with 20-dB isolation and 0.3-dB insertion loss) alternately passes or stops the incoming radiation, producing a square-wave modulation of the signal. (The "off" por-

![Graph](image)

Fig. 6. Intensity of radiation emitted at the electron cyclotron frequency as a function of $n_e$, for hot-ion energies of 8 and 2 keV.

![Diagram](image)

Fig. 5. Schematic diagram of the experimental apparatus for measuring electron temperature.
tion of the square wave corresponds to radiation from the matched load, which is at room temperature.)

The radiation then enters a balanced mixer preamplifier. The electrically tunable local oscillator has a frequency automatically tuned to track the electron cyclotron frequency in a selected portion of the plasma, in spite of unavoidable variations in the current of the confining-field magnet. The gain stability of the amplifying system is improved by feedback from the crystal current of the balanced mixer to control the output power from the local oscillator. Both sidebands of the mixer output are used. The signal then passes through an IF amplifier having a bandwidth of 40 MHz and a center frequency of 60 MHz. The plasma region sampled (neglecting Doppler broadening, etc.) thus is represented by the volumes in the "wall thickness" of two nested closed shells. The wall thickness of each shell corresponds to the 40-MHz bandwidth, and the unsampled volume between the shells corresponds to the 80-MHz separation of the inner edges of the two sidebands. Doppler broadening, which blurs the boundaries of the sampled volume, is assumed to have a negligible over-all effect since the radiation shifted into the accepted bands approximately equals that shifted out. The wide bandwidth of the receiver tends to iron out the effect of variations in the coupling of the horn antenna to the plasma which arise because of cavity resonances in the complicated vacuum chamber.

The detected output from the IF amplifier is modulated at the switching frequency of the latching circulator. The signal is fed into a narrow band ac amplifier and synchronous detector which is phase-locked to the same switching frequency. A low pass filter converts the output from the synchronous detector to a dc signal which is then recorded on magnetic tape.

In the present measurements, for which the plasma is tenuous, the electron cyclotron radiation is proportional to the product of the electron density and the electron temperature. According to Eq. (15), \( T_e \) is proportional to \( (n_i/n_0)^{0.4} \). Therefore, the intensity of the radiation emitted is proportional to \( n_i^{1.4} \). In practice, some variation in the background gas density could not be avoided; however, this dependence is eliminated by dividing the intensity of the emitted electron cyclotron radiation by \( \tau_{ex}^{2/3} \), where \( \tau_{ex} \) is the characteristic decay time due to charge exchange and is inversely proportional to \( n_0 \). Following this procedure, we obtain the results shown in Fig. 6 when the electron radiation is plotted as a function of \( n_i \). The dependence on \( n_i \) for measurements made at ion energies of 8 and 2 keV is in good agreement with the \( n_i^{1.4} \) dependence expected from theory. These results are discussed below at greater length.

### IV. COMPARISON OF THEORY AND EXPERIMENT

#### A. Equilibrium Case

In this section we compare the experimental results with predictions of the simple-model theory of Sec. II, which employs first-order differential equations. Comparison is also made between results of the simple-model theory and those of the Fokker–Planck equations. We concentrate here on the equilibrium situation, i.e., where changes in the plasma density are slow compared with the collisional relaxation time of the electrons.

The outstanding characteristic of the experimental data is that they agree with the easy-to-remember theoretical result that the plasma potential and electron temperature vary as \( (n_i/n_0)^{0.8} \). This behavior as a function of the trapped-ion and background-gas densities was given by Eqs. (15) and (16), approximating the more complete theory represented by Eqs. (6) and (10).

We first discuss the results of the plasma-potential measurements. The experimental method for obtaining these data was presented in Sec. IIIA. Following beam turnoff, the plasma density may be represented by

\[
n_i = n_{eq} \exp\left(-\frac{t}{\tau_{ex}}\right),
\]

(18)

where \( n_{eq} \) is the plasma density at the time of beam turnoff and \( \tau_{ex} \) is the characteristic charge-exchange decay time constant. By substituting Eq. (18) into the relationship \( \phi \propto \left(n_i/n_0\right)^{0.8} \), one obtains the condition that the plasma potential decays with a time constant equal to \( 2.5\tau_{ex} \). This assumes quasiequilibrium conditions and a constant \( n_0 \). Figure 7 shows the experimental behavior of the plasma potential and the fast-proton density during one of our plasma decays. We observe that the potential decreases at a much slower rate after beam turnoff than the ion density, with the ratio of the time constants approaching the predicted value of 2.5. (These data, like the rest of the experimental plasma-
potential results discussed here, were obtained at 15-keV injection energy.)

Several of these decays have been analyzed and the data summarized in Fig. 8. There we show results, both experimental and theoretical, of potential vs \( n_i/n_0 \). The circle at the right end of each individual experimental decay represents the earliest point on the decay after beam turnoff at which the plasma has settled down. After that, the fast-ion and plasma-potential decay times are about constant. We disregard the period directly after beam termination, during which the plasma potential often behaves erratically. The erratic behavior probably is due to the unstable condition of the plasma just before beam turnoff and to the nonequilibrium effect following beam turnoff. This latter effect will be discussed later.

The arrow starting at a given circle and pointing to the lower left shows the measured behavior of the potential during the rest of that plasma decay. The attached arrowhead is positioned at the end of the analyzed data. The numbers by the circle give the background-gas densities in units of \( 10^7 \) molecules/cm\(^3\) for the various examples of plasma decay, assuming hydrogen molecules as the main constituent. These densities range over a factor of 12, from \( 1.3 \times 10^7 \) to \( 1.6 \times 10^8 \) cm\(^3\). The fast-proton densities represented by the data extend over a factor of 58, from \( 8.0 \times 10^6 \) to \( 4.6 \times 10^8 \) ions/cm\(^3\). (All ion densities cited are peak densities.)

The slope of the dashed line in Fig. 8, arbitrarily normalized to fit the data, is that of the predicted \( (n_i/n_0)^{2/5} \) variation. It is in agreement with the general trend of the experimental data. We will make a more quantitative comparison shortly.

Also shown in Fig. 8 are the results predicted by the more complete theory represented by Eqs. (6) and (10), which yield a magnitude as well as a slope. In these calculations we use 15 keV for the proton energy and 0.015 keV for \( E_0 \), the energy with which the electrons are born. Also, we use for \( \sigma \) the value for electrons at energy \( e\Phi \) rather than at some other energy, such as \( kT_e \). From these calculations, one again observes that the predicted slope is in approximate agreement with the data. One also notices that the predicted magnitude of the plasma potential lies close to the plotted experimental values. In general, the agreement between experiment and theory is good. It is better toward the lower left of the plot than might be expected, considering that the uncertainties both in theory and experiment become more dominant there.

We now examine the agreement between the data and theory more quantitatively. From the slope of each plasma decay in Fig. 8, we can calculate a value for the exponent in the power law relating the plasma potential and fast-ion density. The average for the eight experimental values is \( 0.45 \pm 0.06 \), as compared to the basic predicted value of \( \frac{5}{3} \) or 0.40. A value of 0.48 is obtained from the more refined calculation based on Eqs. (6) and (10).

The good agreement here may be somewhat fortuitous because of the various uncertainties involved, such as uncertainty about the composition of the background gas, uncertainties in measuring the experimental ion-density and plasma-potential decay characteristics, and approximations in the theory. Also, the equations were derived for the equilibrium situation, thus representing the plasma during decay only if the time-dependent effects are negligible. The period directly after beam turnoff is not included in these data, to minimize the effects of the nonequilibrium during this period (see Sec. IVB). Despite the uncertainties, the average of the data falls close to theoretical predictions.

In the results presented here, we have emphasized agreement of the data with the \( n_i^{2/5} \) theoretical prediction. From Fig. 8, one can also note the predicted \( n_0^{-2/5} \) variation. The data for the eight decays, although representing a factor-of-12 variation in \( n_0 \), still cluster around the predicted dashed slope. A variation with \( n_0 \) much different from the \( -\frac{2}{5} \) power would not have given this result. Some of the observed data scatter may be due to variations in the background-gas composition, not considered in the calculation of \( n_0 \).

Also shown in Fig. 8 are measurements of the electron temperature (at \( 8 \) keV) taken by the technique in Sec. IIIIB. This technique gives relative and not absolute values. The electron-temperature data are arbitrarily placed high on the plot, so as not to interfere with presentation of the plasma-potential data. Once again, the average slope, indicated by the straight-line fit to the data, is the predicted \( (n_i/n_0)^{2/5} \) behavior, a further experimental verification of the theory.

We now turn to comparing the results of the simple differential-equation model with those of the Fokker–Planck equations. The Fokker–Planck results were obtained at \( 1 \) keV, so the comparison is made at a lower
energy than the 8–15-keV range discussed above. The Fokker–Planck calculations and analytical results are given in Fig. 9. Comparing the electron-temperature calculations from the analytical solutions with those from the Fokker–Planck equations shows reasonable agreement. The plasma-potential results in both cases, as expected, scale as \( (n_i/n_0)^{2/5} \), although the absolute magnitudes are not in as good agreement as for the \( kT_e \) calculations. A difference of this magnitude is not unexpected, probably resulting from the use of average cross sections in the analytical calculations. In the Fokker–Planck equations, the more significant cross sections (e.g., \( \sigma^e \), \( \sigma^i \), and \( \sigma^{re} \)) are treated correctly as a function of energy.

**B. Nonequilibrium Case**

Some important physical processes of interest in high-temperature plasmas are time-dependent or of a nonequilibrium nature. In this section we explain how solutions of the time-dependent Fokker–Planck equations are helpful in interpreting experimental data. We limit our discussion to a specific example, in which a changing electron temperature is a factor influencing the stability of an electrostatic “Harris-type” mode in the plasma.

The behavior of a 15-keV plasma produced by injection and trapping of energetic neutral atoms is shown in Fig. 10 under conditions of strong instability activity. The neutral atom beam is on for approximately 3 sec. During this time the plasma density builds up and becomes strongly unstable. This unstable behavior results in plasma losses correlated with rf activity and large excursions of the plasma potential. The plasma is observed to be stable over most of its decay following beam turnoff. The plasma-potential signal in Fig. 10 is not the slow-ion cutoff signal discussed elsewhere in this article. This electrostatic pickup signal results from the instability, and is much larger than the dc-potential signal.

In an earlier paper, it was suggested that the quiescent period between bursts, and also the stable decay,
might be associated with a change in the electron distribution function. Additional results now indicate that an increase in the electron temperature plays a significant role in stabilizing the plasma during the decay following beam turnoff. One example is the Fokker–Planck results displayed in Fig. 2. The parameters for these calculations were chosen to correspond to the experimental parameters of the plasma produced by the beam pulse of Fig. 10.

As exhibited by the Fokker–Planck calculations of Fig. 2, the average electron energy is seen to increase during the plasma decay following beam turnoff. The time-dependent or nonequilibrium nature of these calculations is clearly indicated by reploting the data from Fig. 2 in the form shown in Fig. 11. The lower curve gives the electron energy during buildup of the plasma density, and the upper curve gives the electron energy during the decay following beam turnoff. The electron energy is not just a function of the ratio \( n_i/n_e \), as in the equilibrium case, but increases following beam turnoff and is significantly higher during plasma decay than during buildup.

An increasing electron temperature following beam turnoff is expected from a consideration of the simple heat balance equation

\[
\begin{align*}
d & \frac{d\langle n_e kT_e \rangle}{dt} = \frac{3}{2} kT_e \frac{dn_e}{dt} + \frac{3}{2} n_e k \frac{dT_e}{dt} \\
& = -[n_e^2(\sigma v v_t) F(\eta/kT_e)](\eta) + n_i n_e G(W, T_e),
\end{align*}
\]

where \( F(\eta/kT_e) \) is defined in Sec. IIIA and \( G(W, T_e) \) is a function of the ion and electron energies. The first term on the right-hand side of Eq. (19) represents the rate at which electrons remove heat from the plasma by escaping over the potential barrier; the second term is the rate at which electrons gain energy by collisions with the ions. In this equation we neglect the energy with which the electrons are born. At equilibrium, the left side of Eq. (19) must be equal to zero since neither \( n_e \) or \( T_e \) are changing with time; therefore, the right side also is zero. Also, the right side is zero immediately after beam turnoff, and remains approximately zero during the initial decay since both the ion–electron transfer rate and the electron loss rate are binary processes. Therefore, the left side is still zero following beam turnoff, and we have

\[
\frac{dT_e}{dt} = -\frac{T_e}{n_e} \frac{dn_e}{dt} = \frac{T_e}{\tau_{ce}},
\]

where \( \tau_{ce} \) is the characteristic charge-exchange time. The electron temperature thus increases temporarily after beam turnoff on a time scale \( \tau_{ce} \).

We also have experimental measurements showing an increase in electron temperature following beam turnoff. Both the plasma density \( n_i \) and the electron temperature \( T_e \) are plotted in Fig. 12 as a function of time following beam turnoff. The electron temperature was obtained from microwave measurements of the emitted radiation at the electron cyclotron frequency, as described in Sec. III. The measured value of \( T_e \) was normalized to a calculated value of \( T_e \) at the time of beam turnoff. Figure 12 shows that \( T_e \) does not increase indefinitely, but begins to decrease after a time following beam turnoff comparable to \( \tau_{ce} \).

Further experimental evidence for the nonequilibrium condition of the plasma directly after beam turnoff is the temporary increase in electrostatic potential frequently observed at that time. Here again, the parallel
energy of the slow ions is the measure of potential. As in the case of $T_e$, the transient increase in potential is followed by the customary gradual decay.

In Fig. 13 we replot the experimental data of Fig. 12 and make a comparison with instability-threshold data. The solid straight line of Fig. 13 gives the average instability threshold for the ion-cyclotron instability observed in the Livermore neutral-injection experiment in terms of the density parameter $\epsilon = (\omega_p/\omega_{ci})^2$ as a function of the ratio $k T_e/W_i$, where $W_i$ is the ion energy.\footnote{22} This instability-threshold line is obtained by reploting the average experimental boundary line given in Fig. 2(a) of Ref. 22 as a function of $k T_e/W_i$ instead of $\epsilon_{ph}/W_i$. A theoretical value for the ratio $k T_e/\epsilon_{ph}$ is used in making this transformation. Individual threshold values are found to deviate from the average experimental boundary line by an amount clearly outside the error assignments. The perpendicular component of the propagation vector $k_x$ is shown to be quantized, with deviations from the average boundary line resulting from $k_x$ taking on different quantized values. The dashed lines of Fig. 13 correspond to the maximum and minimum values observed for $k_x$. We have used the same ratio of $k T_e/\epsilon_{ph}$ in these transformations as in the transformation of the average threshold line.

The triangle at the top in Fig. 13 shows the equilibrium density attained for 8-keV neutral injection. The achieved density lies above the threshold for the instability. Following an instability burst, the electron temperature increases; the point characterizing the plasma (Fig. 13) moves to the right from its equilibrium position toward the average stable threshold line, as indicated by the arrow. After the instability activity, the temperature relaxes back toward its equilibrium value until the plasma again becomes unstable. Following beam turnoff, the plasma then decays, following the curved path indicated on Fig. 13 and reaching the average instability-threshold line at $\epsilon = 0.032$. If the electron temperature varied as expected for equilibrium conditions, i.e., $T_e$ proportional to $(n_e)^{2/3}$, the plasma would decay following a straight line of slope 2.5 and reach the average instability-threshold line at $\epsilon = 0.0074$.

The evidence at this time indicates that the changing electron distribution is a significant factor in determining the stability of the plasma between bursts and during the decay following beam turnoff. The experimental curve in Fig. 13 shows how the increasing electron temperature after beam turnoff enables the plasma quickly to reach the stable region. The electrons may play more of a role in determining stability than indicated by the comparison in Fig. 13, since the criterion for determining stability during decay cannot be the same as that applied to determine the threshold curve. The threshold curve was determined by requiring a total absence of instability bursts for approximately three charge-exchange times, while during decay the plasma would have reached the stable region in a time short compared with $3 \tau_{ce}$.

**V. COLLISIONAL BROADENING OF THE ION DISTRIBUTION FUNCTION**

Plasmas formed in neutral injection experiments by the trapping of neutral atoms in the energy range 5–20 keV have been subject to velocity-space instabilities resulting from the nearly monoenergetic and highly anisotropic nature of the ion distribution function.\footnote{22–25}

An important objective of the Livermore neutral-injection experiment Alice is to reduce the injection energy to a sufficiently low value such that Coulomb collisions broaden the ion distribution function, thereby reducing the driving terms of the instability. A significant broadening of the distribution function is predicted when the charge-exchange time $\tau_{ce}$ equals the Coulomb scattering time, $\tau_s$.\footnote{28} This equality has been used to determine the required beam and vacuum conditions.

Fokker–Planck calculations appropriate to low-energy injection were undertaken to determine the ion and electron distribution functions more accurately in the presence of the source, charge-exchange losses, and scattering losses. The various input parameters correspond to the injection and trapping of neutral hydrogen atoms having an energy of 1 keV. The calculations reported here do not include the effect of charge exchange between protons and beam atoms. This process, which is negligible for the present calculations, tends to sharpen the trapped distribution.

Figure 14 compares the various ion distribution functions. The dashed curve is the analytical loss-cone distri-
VI. CONCLUDING REMARKS

We have described collisional processes in high-temperature plasmas in terms of a self-consistent theory which relates basic plasma parameters such as the electron temperature, ion energy, and plasma potential to the plasma density. A simple model describing plasmas in quasiequilibrium, for which analytical solutions may be obtained, has been shown to give the correct relationship between many plasma parameters. Experimental measurements of the plasma potential $\phi$ and the electron temperature $T_e$ are in good agreement with this simple model. Both $T_e$ and $\phi$ are found to scale with the $\frac{1}{2}$ power of the ratio $n_i/n_e$, as expected.

Fokker–Planck results agree with the simple theory, and the calculations follow the same $\frac{1}{2}$ scaling law under equilibrium conditions. The Fokker–Planck equations were solved to study the nonequilibrium nature of the electron temperature and plasma potential during plasma decay. These calculations indicate that the initial increase in electron temperature following beam turnoff is a significant factor in the stability of the plasma during decay. Electron-temperature measurements have been presented which have the same time-dependent behavior during plasma decay as predicted by the Fokker–Planck calculations.

Numerical solutions of the Fokker–Planck equations have been used to determine the plasma density necessary to approach a loss-cone distribution for the ions. These results give a larger energy spread and end loss than indicated by earlier calculations. This difference is partially explained by degradation of the ion energy, which results from collisions with the electrons. The earlier calculations, in which the ion energy was considered constant, did not take into account this effect.

ACKNOWLEDGMENTS

The authors are grateful to T. K. Fowler and R. F. Post for helpful discussions, and acknowledge the help of Shirley Rompel in writing the numerical program to solve the Fokker–Planck equations.

This work was performed under the auspices of the United States Atomic Energy Commission.

APPENDIX

As discussed in Sec. II B, we have modified the Fokker—Planck equation to include source and loss terms. The collisional loss rate may be written in terms of the distribution function $f_n(v, t)$ for particles of type "a." We use the form given by BenDaniel and Allis, i.e.,

$$\frac{dn_n}{dt} = -(4\pi)^2 \int_0^\infty f_n(v, t)v^2 \times \left( \sum_b \int_0^\infty k_n(v, v') f_b(v', t)v'^2 dv' \right) dv, \quad (A1)$$

The distribution for a mirror ratio of 2 as obtained by BenDaniel and Allis. The equilibrium ion distribution from the Fokker–Planck calculations containing a source and both charge-exchange and scattering losses is shown by a solid line. Using the formula of Post and Damm, we determined the input parameters for the Fokker–Planck problem by the condition that $\tau_e = \tau_{ex}$. The charge-exchange time $\tau_{ex}$ was chosen to be 40 sec and independent of energy. The mirror ratio was 2. The Fokker–Planck results gave a large energy spread and an end-loss rate three times greater than would be expected from the analytical expression for the scattering cross section.

The experimental curve in Fig. 14 indicates the relative spreading of the distribution function in the presence of the instability. This curve was obtained at 15 keV; the velocity is scaled down such that the peak of the distribution occurs at a velocity corresponding to 1 keV. At 1 keV, a significant loss of ion energy to the electrons occurs, resulting in a shift of the distribution peak to a lower velocity (as indicated by the other curves).

The Fokker–Planck calculations of Fig. 14 correspond to an equilibrium ion density of $10^8$ ions/cm$^3$. At low densities, both the equilibrium ion density and the width of the ion distribution function increase as the injected beam current is increased. As a measure of the relative width of the distribution, we use $\Delta v/v_0$, the full-width at half-maximum divided by the velocity at the peak of the distribution. Figure 15 shows how this quantity varies as a function of the equilibrium ion density. Both the loss-cone limit and the spreading produced by a velocity-space instability in the Livermore neutral-injection experiment are indicated on Fig. 15 for comparison. At 1 keV, the relative spreading produced by collisions should be at least as large as the relative instability spreading observed at the higher energies, if the charge-exchange time is as large as 40 sec. (Diagnostics are not available at present for making this measurement.)
where
\[
k_a(v', v') = \Gamma_a \left( \frac{\Lambda_a}{2\pi^2} \right)^{\frac{3}{2}} \begin{cases} 
1 - \frac{1}{2} (v'^2 / v^2) & v' \geq v', \\
\frac{3}{2} (v/v') & v' \leq v',
\end{cases}
\]
(A2)
and
\[
\Lambda_a = (\log_{10} R_{a, eff})^{-1}.
\]
(A3)
The effective mirror ratio, \( R_{a, eff} \), is given by
\[
R_{a, eff} = R[1 \pm (m_e/m_0) (v^2 / \theta^2)]^{-1},
\]
(A4)
except \( R_{a, eff} = 1 \) when Eq. (A4) gives values of \( R_{a, eff} \) less than unity and greater than zero; \( R_{a, eff} = \infty \) for negative values of \( R_{a, eff} \). In Eq. (A4), \( R \) is the true mirror ratio; the sign is chosen to correspond to the sign of the particle being scattered.

The velocity \( v_0 \) is determined by the condition that the electron kinetic energy is just equal to the plasma potential, i.e.,
\[
e\phi = \frac{1}{2} m_e v_0^2.
\]
(A5)
We define the constant \( \Gamma_a \) by the equation
\[
\Gamma_a = (4\pi e^2 / \mu_e) \log D.
\]
(A6)
The quantity \( D \) is the ratio of the Debye length to the classical distance of closest approach. We consider electrons and ions of \( Z = 1 \). We introduce
\[
f_a = (4\pi e^2 / \mu_e) f_a,
\]
where \( K_a \) is determined from the equation
\[
n_a(0) = K_a \int_0^\infty f_a(x, 0) x^2 \, dx,
\]
(A7)
where \( x = v / v_0 \). The constant \( v_0 \) is a characteristic velocity. We introduce the dimensionless variable \( \tau \), where \( \tau = (\frac{1}{2} \Gamma_a K_a / v_0) t \), and define functionals
\[
M(f_a) = \int f_a(y, \tau) y \, dy,
\]
(A8)
\[
N(f_a) = \int f_a(y, \tau) y^2 \, dy,
\]
(A9)
and
\[
E(f_a) = \int f_a(y, \tau) y^4 \, dy.
\]
(A10)
In terms of these new variables, the Fokker–Planck equation for the distribution functions becomes\(^{12}\)
\[
\frac{\partial f_a}{\partial \tau} = A \frac{\partial^2 f_a}{\partial x^2} + B \frac{\partial f_a}{\partial x} + C f_a + D,
\]
where
\[
A = \frac{3}{2} \sum_b K_b / K_a \left[ \frac{1}{3} x^2 \right] E(f_b) + M(f_b),
\]
(B = (4/3x) \sum_b K_b / K_a \left[ \frac{1}{3} x^2 \right] (m_e/m_0) N(f_b)
\]
\[- \left( 1/2 x^2 \right) E(f_b) + M(f_b),\]
\[
C = 2 \left( \frac{m_e}{m_0} \right)^2 \sum_b \left( \frac{m_e}{m_0} \right) \left( \frac{K_b}{K_a} \right) \frac{2 \Lambda_a}{3 x^2} \sum_b K_b - \frac{3}{2 x} \left( \frac{1}{x^2} \right) N(f_b) - \frac{1}{2 x} E(f_b) + M(f_b) - H_a(x, \tau).
\]
The term \( D(x, \tau) \) describes the time-dependent sources and \( H_a(x, \tau) \) is the charge-exchange loss term, which is zero for electrons.

At any time step we can determine the number density and average energy of each type of particle. Let \( I_\theta(\tau) \) and \( I_\phi(\tau) \) be the second and fourth moments of the distribution functions, i.e.,
\[
I_\phi(\tau) = \int_0^\infty f_a(x, \tau) x^2 \, dx,
\]
(A12)
\[
I_\phi(\tau) = \int_0^\infty f_a(x, \tau) x^4 \, dx,
\]
(A13)
The number density is given by
\[
n_a(\tau) = K_a I_\phi(\tau),
\]
(A14)
and the mean energy is given by
\[
E_a(\tau) = \frac{3}{2} k T_a = \frac{3}{2} m_e v_0^2 \left[ I_\phi(\tau) / I_\phi(\tau) \right].
\]
(A15)
We assume that the injected particles have a velocity distribution defined by \( S_a(x) \). The source terms \( D(x, \tau) \) are given by
\[
D(x, \tau) = S_a(x) \left( \frac{1}{\tau} \right) \frac{1}{K_a N(S_a)} \left( \frac{dn_a}{dt} \right),
\]
(A16)
where \( (dn_a/dt) \) is the trapping rate for particles of type “\( a \).” We can include up to 10 sources of the type given by Eq. (A16), corresponding to multiple beam injection at different energies.

In the above discussion of source terms, the \( \sigma \) terms were treated as constants; however, the cross sections have a velocity dependence determined by experimental measurements. We have polynomial descriptions for these functions so the terms \( \sigma v \) terms can be replaced by integrals involving the distributions.

The critical velocity \( x_c \) is determined by the following procedure. At every time step \( n_a(\tau) \) and \( n_i(\tau) \) are computed, and the difference \( | n_a(\tau) - n_i(\tau) | \) is also computed. Since we wish to keep \( n_a(\tau) \) \( \geq n_i(\tau) \), the above difference is compared to a pre-assigned small number. If the difference exceeds this number, then \( x_c \) is increased by an amount \( \Delta x_c \) if \( n_i(\tau) - n_a(\tau) \) is positive, or decreased by an amount \( \Delta x_c \) if \( n_i(\tau) - n_a(\tau) \) is negative. The time step is then repeated. This process is repeated until \( | n_a(\tau) - n_i(\tau) | \) is sufficiently small; the calculation is then continued.

The coupled nonlinear partial differential equations for the functions \( f_a(v, t) \) are solved numerically using
finite-difference methods. The equations are not linearized, i.e., the coefficients involving moments of the distribution functions are computed at each time step. An implicit difference scheme is used, that is, the velocity derivatives are replaced by difference quotients taken at the new time step, while the coefficients are evaluated by use of the distribution function of the previous time step and then extrapolated. The scheme is numerically stable in practice, with no restriction on the time step. This is an essential part of the calculations because as the electron temperature increases, the transfer rate decreases; the time step $\Delta t$ therefore must be increased continually for the calculation to progress toward equilibrium in a sensible manner. A more detailed description of this method for obtaining numerical solutions is given in Ref. 15.

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