

Qualifying Exam for Graduate Students

August 2018

Physics and Astronomy Department, Brigham Young University

Worked Problem Section

Instructions: This section of the qualifying exam requires worked-out answers. It will be worth 2/3 of the total exam. There are 14 topics, of which you must choose eight to answer. The eight topics you choose will be weighted equally. If you work on more than eight of the topics, please indicate clearly which eight you would like to be graded.

The 14 topics are:

- | | |
|---------------------------|------------------------|
| 1. Mathematical Physics 1 | 8. Quantum Mechanics 2 |
| 2. Mathematical Physics 2 | 9. Optics |
| 3. Mechanics | 10. Acoustics 1 |
| 4. Thermodynamics | 11. Acoustics 2 |
| 5. Electrodynamics 1 | 12. Astronomy 1 |
| 6. Electrodynamics 2 | 13. Astronomy 2 |
| 7. Quantum Mechanics 1 | 14. Solid State |

Work each problem on the paper that has been provided. Start each problem on a new piece of paper. When you finish the exam, make sure that all of your work is placed in the appropriate divider sections. You will have four hours for this section. Student calculators are permitted. Some possibly helpful electricity/magnetism equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \nabla \cdot \mathbf{D} = \rho_f \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \mathbf{D} = \epsilon \mathbf{E} \quad \mathbf{M} = \chi_m \mathbf{H} \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} \quad \text{Linear}$$

$$\nabla \cdot \mathbf{P} = \rho_b \quad \mathbf{P} \cdot \hat{\mathbf{n}} = \sigma_b \quad \nabla \times \mathbf{M} = \mathbf{J}_b \quad \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{K}_b$$

$$(\text{Diverg. Thm}) \quad \int \nabla \cdot \mathbf{F} d\tau = \oint \mathbf{F} \cdot d\mathbf{a} \quad (\text{Stokes' Thm}) \quad \int (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint \mathbf{F} \cdot d\mathbf{l}$$

name: _____

Mathematical Physics 1

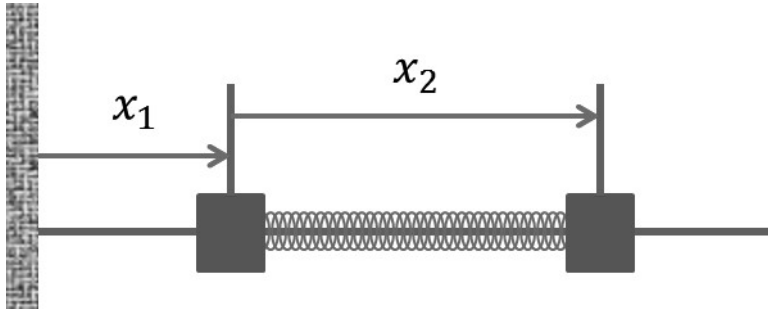
You have a circular disk of radius a , which has both the top and bottom insulated so that no heat flows out of them and there is no z -dependence in the problem. The outer edge is also insulated. The initial temperature distribution is

$$f(r) = \begin{cases} 0 & r < \frac{a}{2} \\ 100 & \frac{a}{2} < r < a \end{cases}$$

- (a) Find the temperature distribution at later times. Use the notation that α_{nj} is the j^{th} zero of the n^{th} Bessel function and α'_{nj} is the j^{th} zero of the *derivative* of the n^{th} Bessel function. Express your coefficients in your series as integrals; don't attempt to perform them. Be sure you consider all the possible values of the separation constant.
- (b) On physical grounds, what do you expect the final state to look like? Can you give a quantitative answer to this? If you can, do so.

Mechanics

Two masses, m_1 and m_2 , are joined by a specially prepared spring that does not obey Hooke's law. The masses slide frictionlessly on a horizontal rod.



The position of m_1 is measured from the wall on the left. The position of m_2 is measured with respect to m_1 as shown above. In terms of these coordinates, the potential energy of the spring is given by the formula

$$U = \frac{1}{4} \alpha (x_2 - \ell)^4$$

where α and ℓ are constants.

- Find the Lagrangian for the system.
- Find the generalized momenta conjugate to x_1 and x_2 .
- Write the equations of motion you obtain from the Lagrangian. You need not solve the equations.
- Write the Hamiltonian for the system.
- Find the equations of motion you obtain from the Hamiltonian.
- If initially $x_1 = 5$ cm, $x_2 = 3$ cm and $\ell = 2$ cm and the system is released from rest, what is the minimum value x_2 can attain? What is the total kinetic energy of the system when x_2 is at its minimum value? Take each mass to be 50 g.

Thermodynamics

Consider a system of N noninteracting magnetic moments with spin $1/2$ in equilibrium with a heat bath at temperature T . The interaction energy of a magnetic moment $\boldsymbol{\mu}$ in a magnetic field \mathbf{B} is $E = -\boldsymbol{\mu} \cdot \mathbf{B}$, which we can write

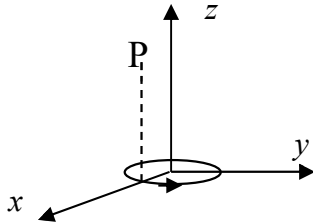
$$E = s\mu B,$$

where $s = \pm 1$ and μ is the z -component of $\boldsymbol{\mu}$.

1. Find the partition function for a single magnetic moment in the magnetic field, Z_1 .
2. Find the partition function for N spins, Z_N .
3. Find the average energy of the magnetic moments.
4. Find the Helmholtz free energy, F , of the system.
5. Estimate the entropy of the system as $T \rightarrow 0$ and $T \rightarrow \infty$.

Electrodynamics 1

Suppose a loop of current I having radius R lies in the x - y plane as shown, centered at the origin. You want to determine the magnetic field at point "P", a distance z above the $(R,0,0)$ edge of the loop.



(a) Qualitatively, in what direction would you expect the field to be at point P? Explain your reasoning.

(b) To actually obtain the field at this point, you could use the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\boldsymbol{\ell}' \times \boldsymbol{r}}{r^3}$$

where \boldsymbol{r} is the vector pointing from source point \mathbf{r}' to the point where you want to know the field, \mathbf{r} :

$$\boldsymbol{r} = \mathbf{r} - \mathbf{r}'$$

Determine the expressions for \boldsymbol{r} , r , and $d\boldsymbol{\ell}'$ that you would need to use to apply this formula and set up the integral. Put all unit vectors in your \boldsymbol{r} and $d\boldsymbol{\ell}'$ answers appropriately in terms of the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ unit vectors. Don't bother doing the $d\boldsymbol{\ell}' \times \boldsymbol{r}$ cross product. Specify the limits of integration that you would need to use in the integral, but don't bother actually doing the integral.

Electrodynamics 2

Imagine that I have two solenoids. They both consist of a single layer of windings, with n windings (turns) per unit length for both coils. One of them is very, very long and has a radius r_a . The other has a larger radius r_b and has a length L . I place the shorter solenoid with the larger radius around the longer, narrower solenoid such that the two share a common axis, and such that the shorter solenoid is centered on the longer one, as shown in the image below.



1. The shorter, larger radius coil is not connected to anything - the leads are left open. A power source drives a current in the longer, slender solenoid which changes in time according to the equation

$$I_a = I_0 \sin(\omega t),$$

where I_0 and ω are constants, and ω is low enough that you don't have to consider radiation, retarded potentials, etc. To within a sign, what is the emf \mathcal{E}_b as a function of time induced in the shorter coil by the current in the longer coil?

2. Now I disconnect the longer coil and use a power source to drive current through the shorter coil which changes in time according to the equation

$$I_b = I_0 \sin(\omega t).$$

To within a sign, what is the emf \mathcal{E}_a induced in the longer coil?

Quantum 1

Generalized uncertainty principle

- a) Using the generalized uncertainty principle provided below, derive the Heisenberg's uncertainty principle for the operators \hat{p} and $f(\hat{x})$ where f is a given function of x .

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle [A, B] \rangle}{2i} \right)^2$$

- b) Calculate the commutator $[\hat{x}, \hat{H}]$ and derive the uncertainty principle for \hat{x} and \hat{H}

- c) Write the Heisenberg equation of motion provided below for $\hat{Q} = \hat{H}$. What does it tell you?

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar} \langle [H, Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

- d) Apply the Heisenberg's equation of motion to the case of $\hat{Q} = \hat{x}$. Show how it is leading to a definition for $\langle \hat{p} \rangle$. From the generalized uncertainty principle, and using $\hat{Q} = \hat{x}$, derive a time – energy uncertainty principle, where you carefully define Δt and ΔE ?

Quantum Mechanics 2

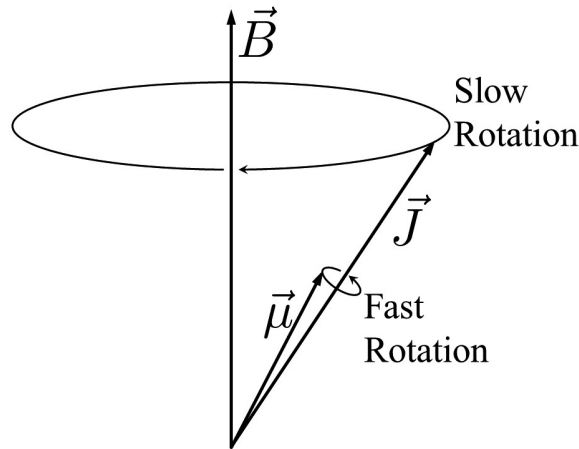
Consider the hydrogen atom at rest in a magnetic field, \vec{B} . The Hamiltonian of the system is written,

$$\hat{H} = \hat{H}_a + \hat{H}', \quad (1)$$

where \hat{H}_a is the unperturbed atomic Hamiltonian and \hat{H}' represents the perturbation due to the magnetic field,

$$\hat{H}' = -\vec{\mu} \cdot \vec{B} = \frac{e}{2m} (\vec{L} + 2\vec{S}) \cdot \vec{B}, \quad (2)$$

where $e = 1.60 \times 10^{-19}$ C and $m = 9.11 \times 10^{-31}$ kg. When the magnetic field is not too strong, the magnetic moment $\vec{\mu}$ precesses rapidly around the total angular momentum \vec{J} . The total angular momentum \vec{J} precesses slowly around the external field, \vec{B} , as suggested in the drawing below.



1. Show that

$$\langle \vec{\mu} \cdot \vec{B} \rangle \simeq \left\langle \frac{(\vec{\mu} \cdot \vec{J})(\vec{J} \cdot \vec{B})}{J^2} \right\rangle, \quad (3)$$

and explain your reasoning.

2. Show that the energy shift due to the magnetic field can be written as

$$\langle \hat{H}' \rangle = \frac{\hbar\Omega}{2} g_L m_j, \quad (4)$$

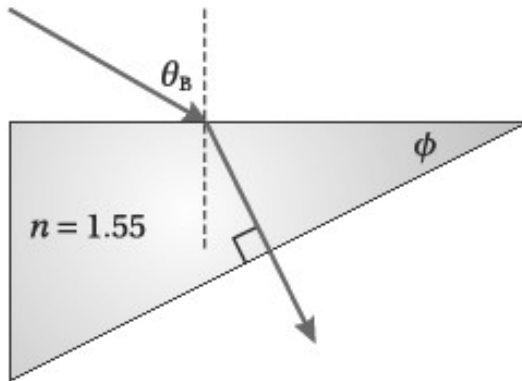
and explain your reasoning.

3. Give an accurate expression for g_L in terms of the quantum numbers j , ℓ , and s .
4. Give an accurate expression for Ω in terms of e , m , and B .

Optics

Light goes through a glass prism with optical index $n = 1.55$. The light enters at Brewster's angle and exits at normal incidence, as shown in the figure.

- (a) Derive and calculate Brewster's angle θ_B .
- (b) Calculate ϕ .
- (c) What percent of p-polarized light (power) goes all the way through the prism?



Hint:

$$n_i \sin \theta_i = n_t \sin \theta_t$$

$$r_p = \frac{E_r^{(p)}}{E_i^{(p)}} = \frac{\cos \theta_i \sin \theta_i - \cos \theta_t \sin \theta_t}{\cos \theta_i \sin \theta_i + \cos \theta_t \sin \theta_t} = \frac{n_t \cos \theta_i - n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$R_p = r_p^2$$

$$R_p + T_p = 1$$

Acoustics 1

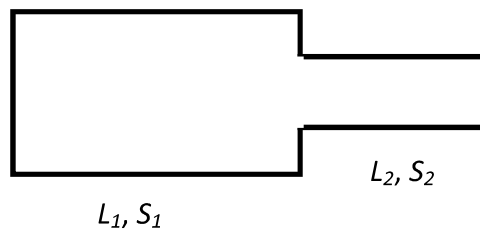
1. In measuring the acoustic response of an ideal 220 Hz sawtooth wave with a microphone, you determine that the fundamental corresponds to a sound pressure level of 72 dB.
 - a. What do you expect the frequencies and sound pressure levels of the next 3 harmonics to be? (As a reminder, the harmonics of a sawtooth wave fall off as $1/n$.)
 - b. What will the overall sound pressure level of the sawtooth wave be?
2. Consider a Helmholtz resonator, characterized by a cavity with length L_1 and cross-sectional area S_1 , along with a neck with length L_2 and cross-sectional area S_2 . (See diagram below.)

- a. Beginning with the impedance translation theorem, given by

$$Z_A = \frac{\rho_0 c Z_{A,L} + j \frac{\rho_0 c}{S} \tan kL}{S \frac{\rho_0 c}{S} + j Z_{A,L} \tan kL} ,$$

develop an expression for the input impedance to the Helmholtz resonator (assume no damping in the system).

- b. If we now assume that all dimensions are small, relative to a wavelength, show that the impedance can be represented as lumped elements and obtain an expression for the resulting lumped-element impedance.



Astronomy 1

Let's look at the region between what we would call the photosphere and the point where the light is finally free to leave the star unimpeded. We have a density structure defined by $\rho = \rho_0(1 + s/R)$ where $\rho_0 = 2.1 \times 10^{-4} \frac{kg}{m^3}$. The opacity is given by $\kappa_0 = 0.03 m^2/kg$ for the continuum. The value of R is related to the scale height in the atmosphere. We will assume that $R = 750 km$. For the H-alpha line let's assume that the opacity is 50 times higher than the continuum level and for H-beta it is 44 times higher. Answer the following questions: A) How deep below the point where the light is free to leave the star would be the 'photosphere' for the continuum and each spectral line? B) Discuss how this would impact our interpretation of the relation between the continuum and the spectral lines. C) How would the answers to (A) change for a Supergiant star where $\rho_0 = 1.1 \times 10^{-5} \frac{kg}{m^3}$.

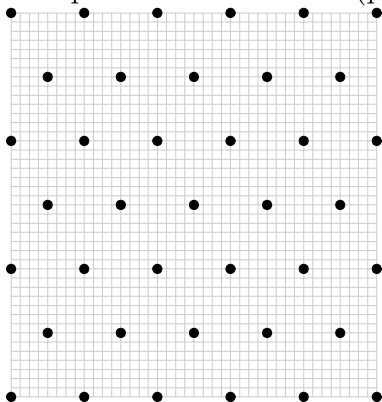
Remember that $\tau = \int_0^s \kappa \rho ds$. I've left off the minus sign so that this integrates to a positive value.

Astronomy 2

Starting from the dust-filled, expanding, Newtonian universe, where the sum of the kinetic energy and potential energy in an expanding Birkhoff sphere is taken to be $-\frac{1}{2}mkc^2\varpi^2$, derive the relationship $[H^2(t) - \frac{8}{3}\pi G\rho]R^2(t) = -kc^2$

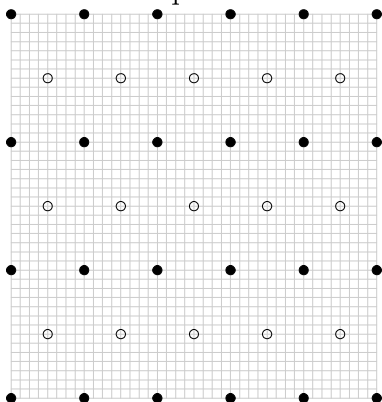
Solid State

1. The picture below shows (part of) a lattice.



- Write down the primitive cell vectors for this lattice.
- Find a set of non-primitive cell vectors for which the unit cell is a rectangle (that is, find a “conventional” unit cell and give its lattice vectors).
- Carefully sketch the Wigner-Seitz cell for this lattice on the diagram.

2. The second picture below shows (part of) a two-dimensional crystal.



- Write down the primitive cell vectors for this lattice and list the atomic basis as well.
- Write down the reciprocal lattice vectors for this crystal.

3.

- Assume a one-dimensional crystal with no crystal potential (i.e., a “free electron”). The lattice parameter is a_0 . Sketch the band structure for the lowest 3 eigenvalues in the first Brillouin zone.
- On the same diagram, draw the band structure assuming there is a small, non-zero crystal potential with a periodicity of a_0 . (Draw in such a way that the grader can tell the difference between the free electron states and states when a weak potential is present.)
- Discuss the physical difference between the lowest two states (or the corresponding charge density) at the zone edge. That is, explain from a physical point of view why the energies are different.