Qualifying Exam for Graduate Students

August 2017

Physics and Astronomy Department, Brigham Young University

Worked Problem Section

Instructions: This section of the qualifying exam requires worked-out answers. It will be worth 2/3 of the total exam. There are 14 topics, of which you must choose eight to answer. The eight topics you choose will be weighted equally. If you work on more than eight of the topics, please indicate clearly which eight you would like to be graded.

The 14 topics are:

1. Mathematical Physics 1
2. Mathematical Physics 2
3. Mechanics
4. Thermodynamics
5. Electricity and Magnetism 1
6. Electricity and Magnetism 2
7. Quantum Mechanics 1
8. Quantum Mechanics 2
9. Optics
10. Acoustics 1
11. Acoustics 2
12. Astronomy 1
13. Astronomy 2
14. Solid State

Work each problem on the paper that has been provided. Start each problem on a new piece of paper. When you finish the exam, make sure that all of your work is placed in the appropriate divider sections. You will have four hours for this section. Student calculators are permitted. Some possibly helpful electricity/magnetism equations:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \qquad \nabla\cdot\mathbf{D}=\rho_f \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B}=0 \qquad \nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{H}=\mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t}$$

$$\mathbf{P}=\epsilon_0\chi_e\mathbf{E} \qquad \mathbf{D}=\epsilon\mathbf{E} \qquad \mathbf{M}=\chi_m\mathbf{H} \qquad \mathbf{H}=\frac{1}{\mu}\mathbf{B}\quad \text{Linear}$$

$$\nabla\cdot\mathbf{P}=\rho_b \qquad \mathbf{P}\cdot\hat{\mathbf{n}}=\sigma_b \qquad \nabla\times\mathbf{M}=\mathbf{J}_b \qquad \mathbf{M}\times\hat{\mathbf{n}}=\mathbf{K}_b$$

$$\text{(Diverg. Thm)}\quad \int\nabla\cdot\mathbf{F}\,d\tau=\oint\mathbf{F}\cdot d\mathbf{a} \qquad \text{(Stokes' Thm)}\quad \int(\nabla\times\mathbf{F})\cdot d\mathbf{a}=\oint\mathbf{F}\cdot dl$$

name: ______________________________

Mathematical Physics 1

I can write any function $f(x)$ on the interval from $0 < x < 1$ in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x).$$

If $f(x) = \cos(3\pi x)$, find an equation for a_n as a function of n. Be careful to note if there are any special values of n for which your equation for a_n is different.

One or more of the following integrals may be useful. If $a \neq b$, then

$$\int_0^1 \sin^2(ax)dx = \frac{1}{2} - \frac{\sin(2a)}{4a}$$

$$\int_0^1 \sin(ax)\sin(bx)dx = \frac{b\cos(b)\sin(a) - a\cos(a)\sin(b)}{a^2 - b^2}$$

$$\int_0^1 \cos^2(ax)dx = \frac{1}{2} + \frac{\cos(a)\sin(a)}{2a}$$

$$\int_0^1 \cos(ax)\cos(bx)dx = \frac{a\cos(b)\sin(a) - b\cos(a)\sin(b)}{a^2 - b^2}$$

$$\int_0^1 \sin(ax)\cos(ax)dx = \frac{\sin^2(a)}{2a}$$

$$\int_0^1 \sin(ax)\cos(bx)dx = \frac{a - a\cos(a)\cos(b) - b\sin(a)\sin(b)}{a^2 - b^2}$$

Mathematical Physics 2

A solid sphere of radius $a = 0.5$ m is partially submerged in a bath of ice water (0° C), while its unsubmerged portion is exposed to room temperature (22° C). Its surface temperature $T_S(\theta)$ is

$$T_S(\theta) = \begin{cases} 22°; & 0 \le \theta < \dfrac{\pi}{3} \\ 0°; & \dfrac{\pi}{3} < \theta \le \pi \end{cases},$$

where θ is the polar angle.

(a) Using only a few brief sentences, describe the partial differential equation, boundary conditions, and symmetry conditions that govern the system. Explain the methods you would use to solve for the steady-state temperature $T(r, \theta, \phi)$ anywhere within the sphere, where r is the radius from its center and ϕ is the azimuthal angle.

The solution has the form

$$T(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta),$$

where

$$A_n = \frac{2n+1}{2a^n} \int_0^{\pi} T_S(\theta) P_n(\cos\theta) \sin\theta \, d\theta.$$

(b) Describe the meaning of A_n and $P_n(\cos\theta)$, and explain why the solution is independent of ϕ.

(c) Using the relationships $P_0(x) = 1, P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_{-(n+1)}(x) = P_n(x)$, and

$$\int_c^1 P_n(x)\, dx = \frac{1}{2n+1}[P_{n-1}(c) - P_{n+1}(c)],$$

compute at least two nonzero coefficients of the series solution.

(d) Find a general form for the coefficients and express the series solution in a compact and efficient form that could be readily implemented in a computer mathematics program to as many terms as desired. It should not involve numerical integration.

Thermodynamics

A large cylindrical silo at an industrial plant contains N atoms of He gas. The silo has height H, cross-sectional area A, and the local gravitational acceleration is $\mathbf{g}$. Assume that the temperature T of the gas is low enough that the variation in the gravitational potential energy in the silo is important, and that the classical ideal gas approximation holds.

1. Use the canonical partition function to find the Helmholtz free energy of the gas, F.
2. Evaluate the derivative of the free energy with respect to A

$$\left(\frac{\partial F}{\partial A}\right)_{H,T,N} \tag{1}$$

   What physical quantity does this derivative represent? What kind of physical process does it relate to? What type of term in the first law of thermodynamics does it represent?

3. Evaluate the derivative of the free energy with respect to H

$$\left(\frac{\partial F}{\partial H}\right)_{A,T,N} \tag{2}$$

   What physical quantity does this derivative represent? What kind of physical process does it relate to? What type of term in the first law of thermodynamics does it represent?

4. Calculate a characteristic temperature T_g, above which gravitational effects on the gas can be ignored.

Hint. The semi-classical partition function for a single particle in a rectangular box is

$$Z_1 = \frac{1}{h^3} \int \cdots \int e^{-\beta E}\, \mathrm{d}p_x\, \mathrm{d}p_y\, \mathrm{d}p_z\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{3}$$

and

$$\int_{-\infty}^{\infty} e^{-ax^2}\, \mathrm{d}x = \sqrt{\frac{\pi}{a}}. \tag{4}$$

Electricity and Magnetism 1

Solve Laplace's equation

$$\nabla^2 V = 0$$

to find the electric potential $V(\mathbf{r})$ for the region between two **coaxial** (infinite) cones with a **common vertex**, one inside the other.

Assume that the interior cone is grounded and the exterior one has a constant potential V_1. Use these as the boundary conditions for Laplace's equation.

Hint: Choose the appropriate coordinate system carefully!

Electricity and Magnetism 2

A charged particle leaving a conducting surface emits radiation. Assuming a planar perfect grounded conductor the radiation can be interpreted as being due to the changing electric dipole moment of the accelerated charge and its **image**. For this problem let the conductor be the x-y plane; the particle is currently at position $(0,0,z)$, with $z > 0$, and is moving perpendicularly away from the plane with positive velocity v and positive acceleration a. The particle has mass m and charge q, with $q > 0$.

(a) Sketch the situation, namely the charge, the conducting plane, and the image. In what direction is the electric dipole?

(b) In which direction(s) will the emitted radiation be largest? In which direction(s) will the emitted radiation be smallest? Indicate your answers on your sketch and explain your reasoning.

(c) Determine the radiated power for the particle in terms of the given quantities.

Helpful formulas:

To lowest order, the electric and magnetic radiation fields from an arbitrary source are given by

$$\mathbf{E} \cong \frac{\mu_0 \ddot{p}}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}},$$

$$\mathbf{B} \cong \frac{\mu_0 \ddot{p}}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}},$$

and the total radiated power is

$$P = \frac{\mu_0}{6\pi c}\ddot{p}^2,$$

where p is the electric dipole moment and $\ddot{p}$ is its second time derivative. θ and ϕ are the usual spherical coordinates as used in physics, with θ being the polar angle and ϕ the azimuthal angle, but measured relative to the direction of $\ddot{\mathbf{p}}$ instead of the normal z-axis (if they should differ).

Quantum Mechanics 1

A proton is moving in a three dimensional potential

$$V(r, \theta, \phi) = \frac{\cos(kr)}{\sqrt{r^2 + a^2}}$$

where r, θ, and ϕ are the usual spherical coordinates with θ measured from the positive z axis and ϕ measured from the positive x axis in the projection of the position onto the x-y plane.

$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

- What are the possible eigenvalues of L_z if the system is in an eigenstate of L^2 with eigenvalue $6\hbar^2$?
- What is the ϕ dependence of the eigenstates of L_z with eigenvalues of $4\hbar$?
- If the particle is in a state with the wave function

  $$\psi(r, \theta, \phi) = \sqrt{\frac{5}{2\pi^5}} f(r, \theta)(\phi - \pi)^2$$

  where

  $$\int_0^\infty r^2\, dr \int_0^\pi |f(r, \theta)|^2 \sin\theta\, d\theta = 1$$
  $$\int_0^{2\pi} (\phi - \pi)^4 d\phi = \frac{2\pi^5}{5}$$

  what is the probability that it has a z component of 0 for its angular momentum?

Possibly helpful integrals

$$\int_0^{2\pi} (\phi - \pi)^2\, d\phi = \frac{2}{3}\pi^3$$
$$\int_0^{2\pi} (\phi - \pi)^4\, d\phi = \frac{2}{5}\pi^5$$
$$\int_0^{2\pi} (\phi - \pi)^2 e^{im\phi}\, d\phi = \frac{2e^{i\pi m}\left((\pi^2 m^2 - 2)\sin(\pi m) + 2\pi m \cos(\pi m)\right)}{m^3}$$
$$\int_0^{2\pi} (\phi - \pi)^2 e^{im\phi}\, d\phi = \frac{2e^{i\pi m}\left(4\pi m\left(\pi^2 m^2 - 6\right)\cos(\pi m) + \left(\pi^4 m^4 - 12\pi^2 m^2 + 24\right)\sin(\pi m)\right)}{m^5}$$

Optics

Horizontally polarized monochromatic light represented by Jones vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ goes through a half wave plate with fast axis at angle θ, represented by Jones matrix

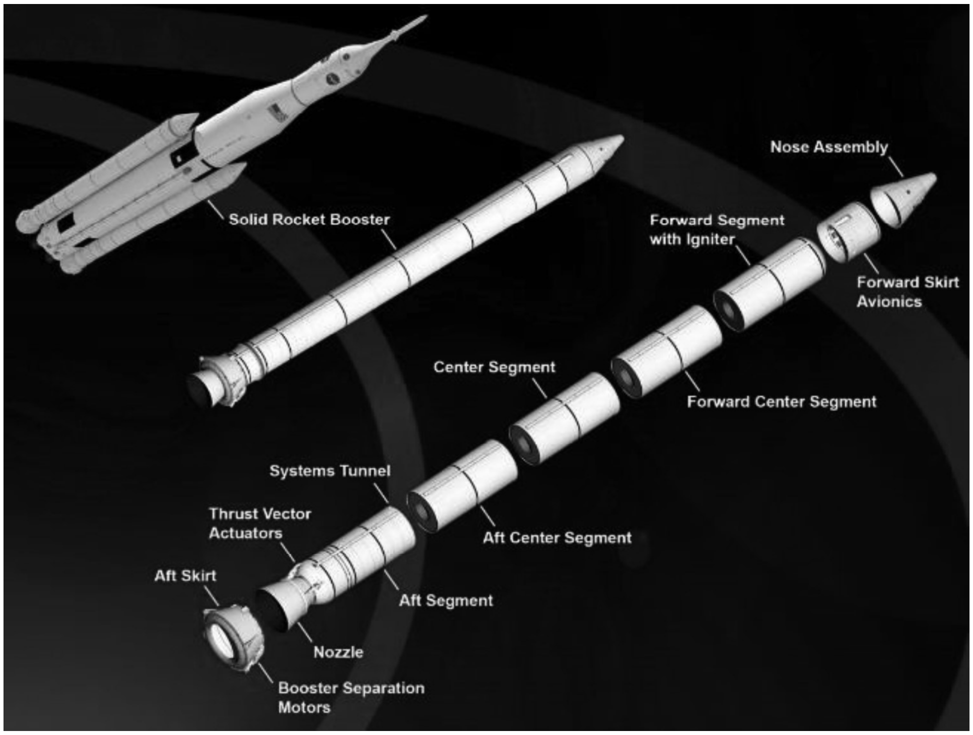
$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

The light then passes through a horizontal polarizer with Jones matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

As a function of wave-plate angle θ, what is the final polarization and intensity of the transmitted light in terms of the intensity of the incident light I_0?

Acoustics 2



1. NASA's Space Launch System designed for human spaceflight beyond lower-earth orbit will have two large solid propellant rocket motors that are similar to, but longer than, the prior Space Shuttle's Space Transportation System. From the SLS booster nozzle to the end of the most forward segment is a nominally cylindrical duct approximately 45 m in length. Given the intense heat of the propellant, consisting in part of burning aluminum particles, the speed of sound inside the motor chamber is approximately 900 m/s.

   a) The acoustic signature inside the pipe produces a low-frequency "organ pipe" resonance that is so strong, it requires the addition of vibration absorbers on the rocket nozzle to prevent the launch vehicle from violently shaking up and down (a phenomenon referred to as "pogo"). These vibration absorbers are relatively simple mass-spring-absorber systems with a tuned natural frequency so as to reduce the vibration on the rest of the launch vehicle. As the acoustics expert on the development team, what frequency do the absorbers need to be tuned to in order to mitigate the effect of the lowest acoustic resonance inside the rocket motor? You may neglect any mean-flow effects.

   b) Consider an acoustic plane wave propagating down the chamber toward the nozzle end. As it reaches the nozzle exit plane, describe (using equations and words) the nominal pressure and particle velocity boundary conditions.

Acoustics 2 continued

c) For this booster, treating the burning rubber, ammonium, and aluminum fuel mixture as heated air gives a reasonable estimate for the temperature inside the chamber. If the speed of sound in air is 343 m/s at 20 deg. C, calculate an estimated motor chamber temperature.

d) Provide an appropriate criterion for the geometric far field if the distance from the nozzle to an observer is r and the dominant noise-producing region of the plume is of length L. Next, if you assume the plume noise source is a line array of incoherent monopoles, sketch the *rate of change* in sound pressure level due to spreading to the side of the plume as a function of r/L. Clearly identify key regimes.

e) Although rocket noise source characterization remains an active research problem, assume that the source can be modeled as a free-field monopole with a radius of 2 m and a frequency of 20 Hz. What is amplitude of the monopole radial velocity if the sound pressure level is 155 dB re 20 μPa at a distance of 65 m? By considering the radial displacement associated with this velocity, comment on the physical reasonableness of your answer. Note that a monopole's pressure amplitude may be written as

$$P = \frac{\rho_0 c k Q}{4\pi r},$$

where ρ_0 is ambient density, c is the sound speed, k is the acoustic wavenumber, Q is the source strength or volume velocity, and r is the distance from the monopole.

Astronomy 1

On the date of the summer solstice find to the nearest 0.1 of a minute (*solar time*)

(a) the elapsed time between sunset on the US-Mexican border directly south of the ESC and sunset at the ESC,

(b) the elapsed time between sunset at the ESC and sunset on the US-Canadian border directly north of the ESC.

Assume that "sunset" refers to the center of the sun's crossing the astronomical horizon, not the apparent horizon. Ignore the significant, but small, effects of atmospheric refraction on your answers. Be sure to express your answers in solar, not sidereal, time units. The following data may or may not be helpful in solving this problem: $\epsilon \equiv$ *obliquity of the ecliptic* $= 23°$ 26' 06", $e \equiv$ *eccentricity of the earth's orbit* $= 0.0167$, $\lambda \equiv$ *longitude of all three sunset locations* $= 111°$ 39' 01"W, $\phi_E \equiv$ *latitude of the ESC* $= +40°$ 14' 50", $\phi_M \equiv$ *latitude of the Mexican border directly south of the ESC* $= +31°$ 30' 59", $\phi_C \equiv$ *latitude of the Canadian border directly north of the ESC* $= +49°$ 00' 00". To solve this problem you will need to use the basic laws of spherical trigonometry. Recall that these are:

Cosine Law

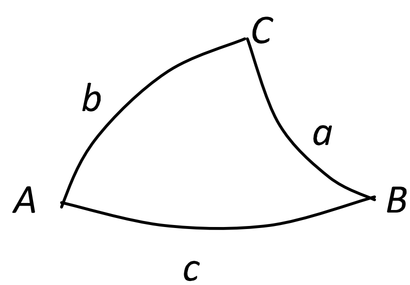
$$\cos a = \cos b \cos c + \cos A \sin b \sin c$$

or

$$\cos A = -\cos B \cos C + \cos a \sin B \sin C$$

Sine Law

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



Five-Parts Formula

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$$

Four-Parts Formula

$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

Astronomy 2

The Na I "D" lines, centered at $\lambda_{D1} = 5895.92\text{Å}$ and $\lambda_{D2} = 5889.95\text{Å}$, have oscillator strengths of $f_1 = 0.33$, $f_2 = 0.66$. In absorption, both arise from a common (ground state) $^2\mathrm{S}_{1/2}$ level. In the questions which follow, the described comparisons are between stars in which spectral absorption lines are not broadened by stellar rotation nor stellar atmospheric turbulence on any scale. Hence the compared line profiles are Voigt profiles. If $A(\lambda)$ represents absorption depth at wavelength λ; W, line equivalent width; λ_0, the line central wavelength; and λ_{fw}, the wavelength at a fixed interval from the central wavelength in the far wings of each line, give the values of

(a) $A_1(\lambda_0)/A_2(\lambda_0)$ in (i) weak-lined stars and (ii) strong-lined stars,

(b) W_1/W_2 in (i) weak-lined stars and (ii) strong-lined stars,

(c) $A_1(\lambda_{fw})/A_2(\lambda_{fw})$ in (i) weak-lined stars and (ii) strong-lined stars.

Solid State

Reciprocal lattice and scattering

Consider a simple cubic structure of lattice size a. Assume that there are four atoms per primitive cell, at the following locations: (0,0,0); (0,1/2,1/2); (1/2,1/2,0) and (1/2,0,1/2).

Assuming the electron concentration around each atom is $n(r) = \frac{1}{(\pi a_0{}^3)} e^{-r/a_0}$ where a_0 is the Bohr radius, compute the form factor f for a given $\vec{\boldsymbol{G}}$ vector in the reciprocal space:

$$f = \iiint dV\, n(\vec{\rho}) e^{-i\vec{G}.\vec{\rho}}$$ where $\vec{\rho}$ is a real-space position vector variable for the integration

Assuming that all these atoms have the same atomic form factor f, compute the structure factor S_G for this crystal, for a given $\vec{\boldsymbol{G}}$ vector in the reciprocal space:

$$S_G = \sum_j f_j e^{-i\vec{G}.\vec{r_j}}$$ where j lists the atoms in the cell